Classification of IIB backgrounds with 28 supersymmetries

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Abstract

We show that all IIB backgrounds with strictly 28 supersymmetries are locally isometric to the plane wave solution of arXiv:hep-th/0206195. Moreover, we demonstrate that all solutions with more than 26 supersymmetries and only 5-form flux are maximally supersymmetric. The N=28 plane wave solution is a superposition of the maximally supersymmetric IIB plane wave with a heterotic string solution. We investigate the propagation of strings in this background, find the spectrum and give the string light-cone Hamiltonian.

1 Introduction

The geometry of backgrounds with a near maximal number of supersymmetries is strongly constrained. The maximally supersymmetric IIB backgrounds have been classified in [1] and they have been found to be locally isometric to Minkowski space, $AdS_5 \times S^5$ [2] and the maximally supersymmetric plane wave [3]. It has also been shown that IIB backgrounds with more than 28 supersymmetries, $N > 28$, are maximally supersymmetric [4, 5], and that IIB backgrounds with more than 24 supersymmetries are locally homogeneous [6]. The latter implies in particular that the 1-form field strength vanishes, $P = 0$. It is also known that there is a plane wave solution in IIB supergravity with non-vanishing 3- and 5-form field strengths which preserves 28 supersymmetries found by Bena and Roiban in [7], see also [8]. So there are IIB backgrounds with strictly 28 supersymmetries which are not locally isometric to the maximally supersymmetric ones.

The main result of this paper is to show that all IIB supergravity backgrounds with strictly 28 supersymmetries are locally isometric to the plane wave in [7]. This will be achieved using the spinorial geometry method of solving Killing spinor equations (KSEs) [9] as adapted to IIB supergravity in [10, 11, 12] and to near maximally supersymmetric backgrounds in [4]. In particular, the gauge symmetry of IIB supergravity will be used to find the canonical form of the normals to the 28 Killing spinors of the background. Then the integrability condition of the gravitino KSE will be solved to reveal that the only solution is that of [7]. The proof is completed by showing that there are no N=28 IIB backgrounds which can arise as discrete quotients of the maximally supersymmetric ones. This establishes the uniqueness of [7], up to discrete identifications, as a IIB solution which preserves strictly 28 supersymmetries.

Another consequence of our analysis is that all $N > 26$ IIB backgrounds with only 5-form flux are maximally supersymmetric. This follows from the observation that if $G = 0$, the $N = 28$ backgrounds are maximally supersymmetric, and from the property that backgrounds with only 5-form flux preserve an even number of supersymmetries.

The main observation that allows our analysis to be carried out is that the Killing spinors of $N = 28$ backgrounds can be expressed in terms of a basis $(\eta_a, i\eta_a)$, where (η_a) are 14 linearly independent spinors over the complex numbers. The algebraic KSE can be easily solved by expressing the 3-form flux G in terms of the normals to the Killing spinors. Then the local part of the proof which involves the solution of the integrability conditions of the gravitino KSE is separated into three different cases labeled by the isotropy group of one of the normal spinors. These isotropy groups are $SU(4) \ltimes \mathbb{R}^8$, $Spin(7) \ltimes \mathbb{R}^8$ and G_2. In both the $SU(4) \ltimes \mathbb{R}^8$ and G_2 cases, all backgrounds that admit $N = 28$ supersymmetry are locally maximally supersymmetric, and so they do not give new solutions. The solution of [7] arises in the $Spin(7) \ltimes \mathbb{R}^8$ case.

The solution of [7] can be interpreted as a superposition of the IIB maximally supersymmetric plane wave [3] with the solution of the heterotic string, see [7] and also [13], which preserves 14 supersymmetries. The latter can be "embedded" into IIB supergravity and in such a case preserves 28 supersymmetries. Using this interpretation, we investigate the propagation of strings on this background. We find that the light-cone Hamiltonian is the sum of harmonic oscillators and compute their frequencies. We find that all directions of the center of mass mode of the string exhibit the same fre-

quency while the different directions of each oscillating mode exhibit two characteristic frequencies.

This paper is organized as follows. In section 2, we give the normals to the Killing spinors and solve the algebraic KSE. In section 3, we state the integrability conditions of the KSEs. In sections 4, 5 and 6, we solve the integrability conditions of the gravitino KSE in the $SU(4) \ltimes \mathbb{R}^8$, $Spin(7) \ltimes \mathbb{R}^8$ and G_2 cases, respectively. In section 7, we investigate the discrete quotients of maximally supersymmetric backgrounds. In section 8, we solve string theory on the $N = 28$ plane wave background, and in section 9 present our conclusions. In appendix A, we choose the normals to the Killing spinors up to gauge transformations. In appendix B, we summarize the integrability condition of the gravitino KSE, and in appendix C we present a part of the analysis for the $Spin(7) \ltimes \mathbb{R}^8$ case.

2 The Algebraic Killing Spinor Equation

2.1 Normal spinors

The main task here is to identify the four normals to the Killing spinors of $N = 28$ backgrounds. In particular, we shall show that the normals are two spinors which are linearly independent over the complex numbers. This in turn will imply that the Killing spinors can be expressed in terms of a basis $(\eta_a, i\eta_a)$, where η_a are 14 spinors linearly independent over the complex numbers.

For this consider the algebraic KSE of IIB supergravity [2, 14, 15]

$$\mathcal{A}\epsilon \equiv P_A \Gamma^A C\epsilon^* + \frac{1}{24} G_{ABC}\Gamma^{ABC}\epsilon = 0 \ , \tag{2.1}$$

where P and G are the 1-form and 3-form field strengths, respectively, and A, B, C are spacetime frame indices. Since all IIB backgrounds with more that 24 supersymmetries are homogeneous, the scalars are constant and $P = 0$. Therefore the algebraic KSE reduces to

$$G_{ABC}\Gamma^{ABC}\epsilon = 0 \ . \tag{2.2}$$

It is clear now that $\mathcal{A}$ is linear over the complex numbers, ie if ϵ is a solution so is $i\epsilon$.

To continue suppose both algebraic and gravitino KSEs of a background admit 28 supersymmetries and let $\epsilon_1, \dots, \epsilon_{28}$ be the Killing spinors. It is required that $\epsilon_1, \dots, \epsilon_{28}$ are *linearly independent* over the *real* numbers. Since the algebraic KSE is linear over the complex numbers, $i\epsilon_1, \dots, i\epsilon_{28}$ are also solutions of the algebraic equation. If one of these additional solutions is linearly independent from $\epsilon_1, \dots, \epsilon_{28}$, over the reals, the dilatino KSE will admit more than 28 solutions. In such a case, we know that the only solution is $G = 0$ [5]. So there are two possibilities to consider for backgrounds that preserve 28 supersymmetries. Either the algebraic KSE admits a basis of 14 linearly independent solutions, $\{\eta_a\}$, over the complex numbers, and so Killing spinors can be written as

$$\epsilon_r = \sum_{a=1}^{14} f_r{}^a \eta_a + \sum_{a=1}^{14} \tilde{f}_r{}^a i\eta_a \ , \quad r = 1, \dots, 28 \ , \tag{2.3}$$

where $(f, \tilde{f})$ is an *invertible* real 28×28 matrix of spacetime functions, or $G = 0$. In the $G = 0$ case, the gravitino KSE also becomes linear over the complex numbers and so the Killing spinors are in pairs $(\epsilon_a, i\epsilon_a)$. So the 28-plane of Killing spinors in both cases is complex. Therefore, the Killing spinors are *normal* to two spinors ν_1 and ν_2 which are *linearly independent over the complex numbers*. We shall use the gauge symmetry of IIB supergravity to find canonical forms for ν_1 and ν_2 and so simplify the choice of the basis (η_a) of the Killing spinors.

2.2 Solution to the algebraic KSE

Assuming that $G \neq 0$, the solution to the algebraic KSE can be expressed in terms of the normals to the Killing spinors. Before we proceed to show this, we take the Killing spinors to be in the positive chirality Weyl representation, $\Delta^+_{\mathbf{16}}$, of $Spin(9,1)$. In such a case, the normals to the Killing spinors, with respect to the Majorana inner product[1] B, lie in the anti-chiral representation $\Delta^-_{\mathbf{16}}$, [4].

To proceed, note that there is an isomorphism $\Lambda^3(\mathbb{R}^{9,1} \otimes \mathbb{C}) \cong \Lambda^2(\mathbb{R}^{16} \otimes \mathbb{C})$ between the complexified 3-forms on $\mathbb{R}^{9,1}$ and the complexified 2-forms on $\mathbb{R}^{16}$. In particular, identify $\mathbb{R}^{16} \otimes \mathbb{C} = \Delta^-_{\mathbf{16}}$ and write

$$G_{A_1A_2A_3} = \frac{1}{2}\lambda_{ij}\, B(\theta^i, \Gamma_{A_1A_2A_3}\theta^j) \ , \tag{2.4}$$

where θ^i is a basis in $\Delta^-_{\mathbf{16}}$. Note that $B(\theta^i, \Gamma^{[3]}\theta^j) = -B(\theta^j, \Gamma^{[3]}\theta^i)$.

Next we have the identity

$$\frac{1}{3!}B(\phi_1, \Gamma_{A_1A_2A_3}\phi_2)\Gamma^{A_1A_2A_3}\phi_3 = 8B(\phi_2,\phi_3)\phi_1 - 8B(\phi_1,\phi_3)\phi_2 \ , \tag{2.5}$$

where $\phi_1, \phi_2 \in \Delta^-_{\mathbf{16}}$ and $\phi_3 \in \Delta^+_{\mathbf{16}}$. Applying this to the algebraic KSE, we have

$$\frac{1}{3!}G_{A_1A_2A_3}\Gamma^{A_1A_2A_3}\eta_a = 8\lambda_{ij}B(\theta^j, \eta_a)\theta^i \ . \tag{2.6}$$

Choosing $\theta^a = B\eta^*_a$, $a = 1, \dots, 14$, $\theta^{15} = \nu_1$ and $\theta^{16} = \nu_2$, we have that the above equation vanishes iff

$$\lambda_{ab} = 0, \qquad \lambda_{15,a} = \lambda_{16,a} = 0 \qquad a, b = 1, \dots, 14 \ . \tag{2.7}$$

It follows that the solution of the algebraic KSE is

$$G_{A_1A_2A_3} = \lambda\, B(\nu_1, \Gamma_{A_1A_2A_3}\nu_2) \tag{2.8}$$

for λ a complex function. Since in the spinorial geometry approach the normal spinors are determined up to gauge transformations, (2.8) can be used to compute G. If $G \neq 0$, then after rescaling one of the normal spinors we can set $\lambda = 1$. For future use observe that

$$\nabla_{A_1}G_{A_2A_3A_4} = B(\nabla_{A_1}\nu_1, \Gamma_{A_2A_3A_4}\nu_2) + B(\nu_1, \Gamma_{A_2A_3A_4}\nabla_{A_1}\nu_2) \ , \tag{2.9}$$

where ∇ is the frame Levi-Civita connection. We shall show that for all $N = 28$ backgrounds, G is parallel.

[1] We use the spinor conventions of [10]. In particular, $B(\theta, \zeta) = \langle B\theta^*, \zeta\rangle$, where $\langle\cdot,\cdot\rangle$ is a Hermitian inner product and $B = \Gamma_{06789}$.

3 Integrability Conditions

To make further progress, we shall investigate the integrability conditions of the KSEs

$$\begin{aligned} \mathcal{D}_M\epsilon &\equiv \nabla_M\epsilon + \frac{i}{48}\Gamma^{N_1\dots N_4}F_{N_1\dots N_4M}\epsilon - \frac{1}{96}(\Gamma_M{}^{N_1N_2N_3}G_{N_1N_2N_3} - 9\Gamma^{N_1N_2}G_{MN_1N_2})C\epsilon^* = 0\ , \\ \mathcal{A}\epsilon &\equiv G_{M_1M_2M_3}\Gamma^{M_1M_2M_3}\epsilon = 0\ , \end{aligned} \tag{3.1}$$

where we have set $P = 0$ as we have already explained. Since the matrix $(f, \tilde{f})$ in (2.3) is invertible, the integrability conditions on the Killing spinors can be evaluated on the basis $(\eta_a, i\eta_a)$. Because of the complex nature of this basis, as we shall demonstrate, the integrability conditions factorize.

First, we take the ∇-derivative of the algebraic KSE and then substitute for $\nabla\epsilon$ using the gravitino KSE to find

$$\begin{aligned} &\big(\nabla_M G_{N_1N_2N_3}\Gamma^{N_1N_2N_3} - \frac{i}{2}G_{N_1N_2L}F_M{}^L{}_{N_3N_4N_5}\Gamma^{N_1N_2N_3N_4N_5} + iG_{N_1N_2N_3}F_M{}^{N_1N_2N_3}{}_L\Gamma^L\big)\epsilon \\ &\quad -\frac{1}{96}\big[\Gamma_M(G_{N_1N_2N_3}\Gamma^{N_1N_2N_3})(G_{N_4N_5N_6}\Gamma^{N_4N_5N_6}) + 6G_{ML_1L_2}\Gamma^{L_1L_2}(G_{N_1N_2N_3}\Gamma^{N_1N_2N_3}) \\ &\qquad\qquad +144G_{N_1N_2L}G_{N_3M}{}^L\Gamma^{N_1N_2N_3}\big]C\epsilon^* = 0\ . \end{aligned} \tag{3.2}$$

Evaluating this condition on the Killing spinor (2.3) basis $(\eta_a, i\eta_a)$, observe that it factorizes as

$$\begin{aligned} &\big(\nabla_M G_{N_1N_2N_3}\Gamma^{N_1N_2N_3} - \frac{i}{2}G_{N_1N_2L}F_M{}^L{}_{N_3N_4N_5}\Gamma^{N_1N_2N_3N_4N_5} \\ &\qquad +iG_{N_1N_2N_3}F_M{}^{N_1N_2N_3}{}_L\Gamma^L\big)\eta_a = 0\ , \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &\big[\Gamma_M(G_{N_1N_2N_3}\Gamma^{N_1N_2N_3})(G_{N_4N_5N_6}\Gamma^{N_4N_5N_6}) + 6G_{ML_1L_2}\Gamma^{L_1L_2}(G_{N_1N_2N_3}\Gamma^{N_1N_2N_3}) \\ &\qquad +144G_{N_1N_2L}G_{N_3M}{}^L\Gamma^{N_1N_2N_3}\big]C\eta_a^* = 0\ , \end{aligned} \tag{3.4}$$

for $a = 1, \dots, 14$.

In addition, the gravitino KSE integrability condition,

$$[\mathcal{D}_N, \mathcal{D}_M]\epsilon \equiv \mathcal{R}_{NM}\epsilon = 2\mathcal{S}\epsilon - 2\mathcal{T}C\epsilon^*\ , \tag{3.5}$$

implies that

$$\mathcal{S}\eta_a = 0\ , \tag{3.6}$$

and

$$\mathcal{T}C\eta_a^* = 0\ , \tag{3.7}$$

where $\mathcal{S}$ and $\mathcal{T}$ are given in [16] and the special case $P = 0$ that applies here is stated in appendix B for convenience.

In what follows, we shall investigate the above integrability conditions for the various choices of Killing spinors which are specified by choosing their normals up to gauge transformations. It is convenient to label the various cases with the isotropy group of the first normal in the $Spin(9,1)$ gauge group.

4 $SU(4) \ltimes \mathbb{R}^8$

4.1 Normal spinors

A representative for the first $SU(4) \ltimes \mathbb{R}^8$-invariant normal [4, 5] is

$$\nu_1 = -pe_5 - qe_{12345} \ , \tag{4.1}$$

where p, q are complex functions with $|p| \neq |q|$. Observe that if $|p| = |q|$, then ν_1 is $Spin(7) \ltimes \mathbb{R}^8$-invariant and this case will be examined separately. To choose the second normal ν_2 in the $SU(4) \ltimes \mathbb{R}^8$ case, one has to decompose $\Delta^-_{\mathbf{16}}$ under $SU(4) \ltimes \mathbb{R}^8$ and choose representatives for the various orbits, see appendix A. As is mentioned in appendix A the choice of the second normal can be simplified by assuming that the 1-form bilinear of any linear combination of ν_1 and ν_2 is null. This is because if a direction in the (ν_1, ν_2)-plane is associated with a time-like 1-form bilinear, then the corresponding solutions are special cases of G_2 backgrounds we shall analyze in section 6.

To summarize the detailed analysis in appendix A, there are three choices for the normals. These are

$$\nu_1 = -pe_5 - qe_{12345}, \qquad \nu_2 = -ye_{12345} - u^1 e_1 - we_{234}, \tag{4.2}$$

with $\bar{w}p + u^1 \bar{q} = 0$ and $p, w, q, u^1 \neq 0$, where $\bar{w}$ is the complex conjugate of w and similarly for the other functions, or

$$\nu_1 = e_5, \qquad \nu_2 = c\ e^1 \qquad (c \neq 0) \ , \tag{4.3}$$

or

$$\nu_1 = -pe_5 - qe_{12345}, \qquad \nu_2 = -xe_5 - ye_{12345} - c_1 e_{145} - c_2 e_{235} \ . \tag{4.4}$$

4.2 Solutions with $G \neq 0$

Here we shall solve the integrability conditions for all the three choices of normals.

4.2.1 $\nu_1 = pe_5 - qe_{12345}$, $\nu_2 = -ye_{12345} - u^1 e_1 - we_{234}$

This choice of normal leads to a basis (η_a) in the space of Killing spinors which includes the spinors

$$\{e_{1235}, e_{1245}, e_{25}, e_{35}, e_{45}, e_{13}, e_{23}, e_{24}, e_{34}, e_{1345}\} \ . \tag{4.5}$$

Substituting each of these spinors in the integrability condition (3.4) and assuming that $G \neq 0$, one finds that $w = 0$. This is a contradiction because for this choice of normals $w \neq 0$. Hence, there are no solutions unless $G = 0$ which will be considered separately.

4.2.2 $\nu_1 = e_5$, $\nu_2 = c\, e^1$

For this choice of normals, (3.4) is automatically satisfied. To proceed further, consider applying (3.3) and (3.7) to the spinors orthogonal to $\nu_1 = e_5, \nu_2 = ce_1$. These integrability conditions imply

$$F = 0 \ . \tag{4.6}$$

On the other hand using (2.8), the 3-form G can be written as

$$\begin{aligned} G \ = \ & c\big(- e^2 \wedge e^3 \wedge e^4 + e^2 \wedge e^8 \wedge e^9 - e^3 \wedge e^7 \wedge e^9 + e^4 \wedge e^7 \wedge e^8 \\ & - ie^2 \wedge e^3 \wedge e^9 + ie^2 \wedge e^4 \wedge e^8 - ie^3 \wedge e^4 \wedge e^7 + ie^7 \wedge e^8 \wedge e^9 \big) \ , \end{aligned} \tag{4.7}$$

and so

$$G \wedge G^* = 8i|c|^2\, e^2 \wedge e^3 \wedge e^4 \wedge e^7 \wedge e^8 \wedge e^9 \tag{4.8}$$

which does not vanish for $c \neq 0$. Since $F = 0$, these data are incompatible with the Bianchi identity of F for which dF is proportional to (4.8). Hence, there are no solutions unless $G = 0$ which will be considered separately.

4.2.3 $\nu_1 = -pe_5 - qe_{12345}$, $\nu_2 = -xe_5 - ye_{12345} - c_1 e_{145} - c_2 e_{235}$

There are a number of cases to consider. First, if $c_1 = c_2 = 0$ and insisting that ν_1 and ν_2 are linearly independent, then a basis in the (ν_1, ν_2)-plane can be chosen such that the first normal spinor is $Spin(7) \ltimes \mathbb{R}^8$ invariant. Therefore this is a special case of backgrounds with a $Spin(7) \ltimes \mathbb{R}^8$-invariant normal which will be examined separately.

Second, if one of c_1 or c_2 does not vanish, without loss of generality, one can take $c_1 \neq 0$. By applying a $SU(4)$ transformation, we can take $\frac{c_2}{c_1}$ to be a *real* function. In such a case, a basis (η_a) in the space of Killing spinors can be chosen to include the 13 spinors

$$\{e_{15}, e_{25}, e_{35}, e_{45}, e_{12}, e_{13}, e_{24}, e_{34}, e_{1235}, e_{1245}, e_{1345}, e_{2345}, \frac{c_2}{c_1} e_{23} - e_{14}\} \ . \tag{4.9}$$

Substituting this into (3.4), we find the relations

$$pq(c_2^2 - c_1^2) = 0, \quad q(yp - xq)(c_2^2 - c_1^2) = 0, \quad p(yp - xq)(c_2^2 - c_1^2) = 0 \ . \tag{4.10}$$

The solution of the above relations leads to three further sub-cases:

(i) $c_2 \neq \pm c_1$, $p = x = 0$ and $q \neq 0$, which gives

$$\nu_1 = e_{12345}, \quad \nu_2 = -c_1 e_{145} - c_2 e_{235} \tag{4.11}$$

(ii) $c_2 \neq \pm c_1$, $q = y = 0$ and $p \neq 0$, which gives

$$\nu_1 = e_5, \quad \nu_2 = -c_1 e_{145} - c_2 e_{235} \tag{4.12}$$

(iii) $c_2 = \pm c_1$. After a $SU(4)$ transformation to set $c_1 = c_2$ and then re-scaling of ν_2, one finds

$$\nu_1 = -pe_5 - qe_{12345}\,, \quad \nu_2 = -xe_5 - ye_{12345} - e_{145} - e_{235} \tag{4.13}$$

Further simplification of the above three cases is possible by applying (3.4) to the 14-th basis element

$$\eta_{14} = p1 - qe_{1234} + \frac{(py - qx)}{c_1} e_{23} \tag{4.14}$$

in the space of Killing spinors.

In particular, for the $c_2 = c_1 = 1$ case, one obtains

$$(|p|^2 - |q|^2)(2|p|^2 + 2|q|^2 + |yp - xq|^2) = 0\ . \tag{4.15}$$

Therefore, $|p| = |q|$, and thus this solution is a special case of those for which ν_1 is $Spin(7) \ltimes \mathbb{R}^8$ invariant.

For the other two cases for which $c_1 \neq \pm c_2$, (3.4) evaluated on η_{14} implies that $c_2 = 0$. By using the gauge transformation $e^{\frac{\pi}{2}(\Gamma_{12}+\Gamma_{34})}$ (with real basis indices), together with appropriately chosen $SU(4)$ gauge transformations, one can simplify the normals as

$$\nu_1 = e_5, \qquad \nu_2 = c\, e_{345}\ . \tag{4.16}$$

To summarize so far, after solving (3.4), the only choice of normals allowed provided one of them is $SU(4) \ltimes \mathbb{R}^8$-invariant is given in (4.16).

Next let us turn to investigate the remaining integrability conditions for the Killing spinors normal to (4.16). To proceed, we use (2.8) to write G as

$$G = \sqrt{2}\, c\, e^+ \wedge \left(e^1 \wedge e^2 - e^6 \wedge e^7 + ie^1 \wedge e^7 - ie^2 \wedge e^6\right)\ . \tag{4.17}$$

A basis (η_a) for the Killing spinors normal to (4.16) is

$$\{1, e_{15}, e_{25}, e_{35}, e_{45}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}, e_{1235}, e_{1245}, e_{1345}, e_{2345}\}\ . \tag{4.18}$$

Then (3.3) and (3.7) imply that

$$F = 0\ , \quad \nabla G = 0\ , \tag{4.19}$$

ie G is parallel with respect to the Levi-Civita connection.

It remains to solve the integrability condition $\mathcal{S}\eta_a = 0$. Since G is null (4.17), the terms G-quadratic terms in $\mathcal{S}$ can be simplified to write

$$\begin{aligned}\mathcal{S}_{NM} \;&=\; \frac{1}{8}R_{NM,L_1L_2}\Gamma^{L_1L_2} + \frac{1}{32}\big(-\frac{1}{8}\Gamma^{L_1L_2}G_{NM}{}^{Q}G^*_{L_1L_2Q} - \frac{13}{12}\Gamma^{L_1L_2}G^*_{NM}{}^{Q}G_{L_1L_2Q} \\ &-\; \frac{1}{48}\Gamma^{L_1L_2L_3L_4}G_{NML_1}G^*_{L_2L_3L_4} + \frac{1}{4}\Gamma_{[N|}{}^{L_1L_2L_3}G_{|M]L_1}{}^{Q}G^*_{L_2L_3Q}\big)\ .\end{aligned} \tag{4.20}$$

To proceed, if $M = \tilde{M}$, $N = \tilde{N}$, where $\tilde{M}$, $\tilde{N}$ take all values except for "+", and if $N = +$, $M = -$, then we obtain

$$R_{\tilde{N}\tilde{M},L_1L_2}\Gamma^{L_1L_2}\eta_a = 0\ , \quad R_{+-,L_1L_2}\Gamma^{L_1L_2}\eta_a = 0\ . \tag{4.21}$$

On applying $C*$ to both these conditions, we find that

$$R_{\tilde{N}\tilde{M},L_1L_2}\Gamma^{L_1L_2}\eta=0, \qquad R_{+-,L_1L_2}\Gamma^{L_1L_2}\eta=0 \tag{4.22}$$

for all Majorana-Weyl spinors η, which in turn implies that the associated Riemann curvature components vanish. Hence the only non-vanishing components of the Riemann tensor are R_{+i+j} for $i,j=1,2,3,4,6,7,8,9$.

To continue, it is convenient to rewrite the remaining $\mathcal{S}$ integrability conditions as

$$\big(\frac{1}{2}(T^2_{MN})_{L_1L_2}\Gamma^{L_1L_2}+\frac{1}{24}(T^4_{MN})_{L_1L_2L_3L_4}\Gamma^{L_1L_2L_3L_4}\big)\eta_a=0\ , \tag{4.23}$$

where

$$(T^2_{MN})_{L_1L_2}=R_{NM,L_1L_2}-\frac{1}{32}G_{NM}{}^QG^*_{L_1L_2Q}-\frac{15}{64}G^*_{NM}{}^QG_{L_1L_2Q}\ , \tag{4.24}$$

and

$$(T^4_{MN})_{L_1L_2L_3L_4}=-\frac{1}{16}G_{NM[L_1}G^*_{L_2L_3L_4]}+\frac{3}{8}\delta_{N[L_1}G_{|M|L_2}{}^QG^*_{L_3L_4]Q}-\frac{3}{8}\delta_{M[L_1}G_{|N|L_2}{}^QG^*_{L_3L_4]Q}\ . \tag{4.25}$$

It is straightforward to show that the only non-vanishing components of T^2 and T^4 are $(T^2_{+i})_{+j}$, $(T^4_{+i})_{+q_1q_2q_3}$. Using this, the integrability condition $\mathcal{S}\eta_a=0$ is equivalent to

$$\big((T^2_{+i})_{+j}\Gamma^j+\frac{1}{6}(T^4_{+i})_{+j_1j_2j_3}\Gamma^{j_1j_2j_3}\big)\chi_a=0 \tag{4.26}$$

for $\chi_a\in\{e_5,e_{135},e_{145},e_{235},e_{245},e_{345}\}$. In order to analyse the conditions imposed by these integrability conditons we have used a computer assisted computation (CAC)[2]. One finds that $c=0$, however this is a contradiction, since we have assumed $G\neq 0$. In conclusion, in all cases, we deduce that we should take $G=0$.

4.3 Solutions with $G=0$

To investigate the solutions with $G=0$, we write the gravitino integrability condition as

$$\mathcal{S}\eta_a\equiv\left(\frac{1}{2}(T^2_{MN})_{N_1N_2}\Gamma^{N_1N_2}+\frac{1}{4!}(T^4_{MN})_{N_1N_2N_3N_4}\Gamma^{N_1N_2N_3N_4}\right)\eta_a=0 \tag{4.27}$$

where now

$$\begin{aligned}(T^2_{MN})_{P_1P_2} &= \tfrac{1}{4}R_{MN,P_1P_2}-\tfrac{1}{12}F_{M[P_1}{}^{Q_1Q_2Q_3}F_{|N|P_2]Q_1Q_2Q_3}\ ,\\ (T^4_{MN})_{P_1\dots P_4} &= \tfrac{i}{2}D_{[M}F_{N]P_1\dots P_4}+\tfrac{1}{2}F_{MNQ_1Q_2[P_1}F_{P_2P_3P_4]}{}^{Q_1Q_2}\ .\end{aligned} \tag{4.28}$$

The field equations and Bianchi identities imply that

$$\begin{aligned}(T^2_{MN})_{P_1P_2} &= (T^2_{P_1P_2})_{MN}\ ,\quad (T^2_{M[P_1})_{P_2P_3]}=(T^2_{MN})_P{}^N=0\ ,\\ (T^4_{[P_1P_2})_{P_3P_4P_5P_6]} &= (T^4_{MN})_{P_1P_2P_3}{}^N=0\ ,\quad (T^4{}_{P_1(M})_{N)P_2P_3P_4}=(T^4{}_{[P_1|(M})_{N)|P_2P_3P_4]}\ ,\end{aligned}$$

[2]We can provide more information on request.

$$(T^4_{M[P_1})_{P_2P_3P_4P_5]} = -\frac{1}{5!}\epsilon_{P_1P_2P_3P_4P_5}{}^{Q_1Q_2Q_3Q_4Q_5}(T^4_{M[Q_1})_{Q_2Q_3Q_4Q_5]} \ . \tag{4.29}$$

In the $SU(4)\ltimes\mathbb{R}^8$ case, there are two choices of normal spinors that we should consider up to $Spin(9,1)$ transformations. First consider the case in which the two normals can be chosen as

$$\nu_1 = -pe_5 - qe_{12345}, \qquad \nu_2 = -ye_{12345} - u^1e_1 - u^2e_2 - we_{234} - c_3e_{235} \ , -c_4e_{345} \ , \tag{4.30}$$

and $|u^1|^2+|u^2|^2 \neq 0$. This case can be further separated, as in the analysis of the previous section, into two different sub-cases using the additional condition that the associated 1-form bi-linears of all directions in the (ν_1,ν_2)-plane are null, see appendix A. However, there is no advantage to do this here and so we shall treat both sub-cases together. The basis (η_a) of Killing spinors normal to (ν_1,ν_2) in (4.30) includes the spinors

$$\{e_{1235}, e_{1245}, e_{25}, e_{35}, e_{45}, e_{13}, e_{23}, e_{24}, e_{34}, we_{1345}+u^2e_{15}, we_{2345}-u^1e_{15}\} \ . \tag{4.31}$$

Substituting these 11 spinors into (4.27) and making use of (4.29), one obtains $T^2 = T^4 = 0$, so these solutions are locally maximally supersymmetric. Here and in two similar cases below, we have again used CAC.

Next, consider the case for which

$$\nu_1 = -pe_5 - qe_{12345}, \qquad \nu_2 = -xe_5 - ye_{12345} - c_1e_{145} - c_2e_{235} \ . \tag{4.32}$$

To proceed further, it is convenient to in addition assume that $|c_1|^2 + |c_2|^2 \neq 0$. In such a case, the basis (η_a) of Killing spinors includes

$$\{e_{15}, e_{25}, e_{35}, e_{45}, e_{12}, e_{13}, e_{24}, e_{34}, e_{1235}, e_{1245}, e_{1345}, e_{2345}, c_2e_{23}-c_1e_{14}\} \ . \tag{4.33}$$

Substituting these 13 spinors in (4.27) and using (4.29), one finds $T^2 = T^4 = 0$. Again, these solutions are locally maximally supersymmetric. It remains to consider the case for which $c_1 = c_2 = 0$ in (4.32). Then a basis in the space of Killing spinors is

$$(\eta_a) = \{e_{15}, e_{25}, e_{35}, e_{45}, e_{12}, e_{13}, e_{24}, e_{34}, e_{1235}, e_{1245}, e_{1345}, e_{2345}, e_{23}, e_{14}\} \ . \tag{4.34}$$

Substituting this basis into (4.27) and using (4.29), one finds that $T^2 = T^4 = 0$. So the solutions are again locally maximally supersymmetric.

To summarize, if one of the two normal spinors of backgrounds preserving $N = 28$ supersymmetries is $SU(4)\ltimes\mathbb{R}^8$-invariant, then they are locally maximally supersymmetric. Later we shall show that there are no quotients of maximally supersymmetric backgrounds preserving 28 supersymmetries. As a consequence, all such $N = 28$ supersymmetric backgrounds are maximally supersymmetric.

5 $Spin(7)\ltimes\mathbb{R}^8$-invariant normal

5.1 Solutions with $G \neq 0$

It is explained in appendix A that the two normal spinors can be chosen as

$$\nu_1 = e_5 + e_{12345}, \quad \nu_2 = c(e_5 - e_{12345}), \qquad (c \neq 0) \ . \tag{5.1}$$

Using this and (2.8), one finds that

$$G = 2\sqrt{2}\, i\, c\, e^+ \wedge \omega \ , \qquad \omega = e^1 \wedge e^6 + e^2 \wedge e^7 + e^3 \wedge e^8 + e^4 \wedge e^9 \ . \tag{5.2}$$

To proceed, a basis in the space of Killing spinors normal to (ν_1, ν_2) given in (5.1) is

$$(\eta_a) = \{e_{15}, e_{25}, e_{35}, e_{45}, e_{1235}, e_{1245}, e_{1345}, e_{2345}, e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\} \ . \tag{5.3}$$

Substituting this into the integrability conditions (3.3) and (3.7) and after some CAC, one finds that

$$\nabla(c\, e^+) = 0 \ , \quad \nabla G = 0 \ , \quad F = f\, e^+ \wedge \omega \wedge \Psi \ , \tag{5.4}$$

where $f = |c|$ and Ψ is a (1,1)- and ω-traceless form in the directions 12346789 transverse to the light-cone, ie

$$\Psi_{kl}\, \omega^k{}_i \omega^l{}_j = \Psi_{ij} \ , \quad \Psi_{ij}\omega^{ij} = 0 \ . \tag{5.5}$$

Thus $c\, e^+$ and G are ∇*-parallel*. In particular, as (Re c)e^+ and (Im c)e^+ are both covariantly constant, this implies that there exists a *constant angle* ϕ such that $c = f e^{i\phi}$. Hence, the spacetime admits a covariantly constant real 1-form $V = f e^+$. Thus, the spacetime geometry is that of a pp-wave.

It remains to evaluate the last integrability condition $\mathcal{S}\eta_a = 0$, (3.6), on the basis (5.3) of the Killing spinors. The expression for $\mathcal{S}$ can be considerably simplified by making use of the special form for F and G which we have obtained in (5.4) and (5.2), respectively. In particular, one can write

$$\begin{aligned}\mathcal{S}\eta_a \equiv \Big(\frac{1}{8} R_{NM,L_1L_2}\Gamma^{L_1L_2} - \frac{i}{48}\Gamma^{L_1L_2L_3L_4} D_{[N}F_{M]L_1L_2L_3L_4} - \frac{1}{24}\Gamma^{L_1L_2} F_{[N|L_1}{}^{Q_1Q_2Q_3} F_{|M]L_2Q_1Q_2Q_3} \\ + \frac{1}{48}\Gamma^{L_1L_2L_3L_4} F_{NML_1}{}^{Q_1Q_2} F_{L_2L_3L_4Q_1Q_2} - \frac{1}{32}\Gamma^{L_1L_2} G_{NM}{}^{L_3} G^*_{L_1L_2L_3}\Big)\eta_a = 0 \ .\end{aligned} \tag{5.6}$$

Next observe that the 14-plane spanned by the basis (5.3) of the Killing spinors is invariant under the reality operation $C*$, ie $C * \{\eta_a\} = \{\eta_a\}$. Moreover, using that $G = i f e^{i\phi} H$, where H is a *real* 3-form, which in turn implies that the G-quadratic terms in $\mathcal{S}$ are real, one finds that $\mathcal{S}\eta_a = 0$ factorizes as

$$\Gamma^{L_1L_2L_3L_4} D_{[N}F_{M]L_1L_2L_3L_4}\eta_a = 0 \ , \tag{5.7}$$

and

$$\begin{aligned}\Big(\big(\frac{1}{2} R_{NM,L_1L_2} - \frac{1}{8} G_{NM}{}^Q G^*_{L_1L_2Q} - \frac{1}{6} F_{[N|L_1}{}^{Q_1Q_2Q_3} F_{|M]L_2Q_1Q_2Q_3}\big)\Gamma^{L_1L_2} \\ + \frac{1}{12} F_{NML_1}{}^{Q_1Q_2} F_{L_2L_3L_4Q_1Q_2}\Gamma^{L_1L_2L_3L_4}\Big)\eta_a = 0 \ .\end{aligned} \tag{5.8}$$

Let us first focus on (5.8). Setting $N=\hat{N}$ and $M=\hat{M}$, where $\hat{N}$ and $\hat{M}$ take all values apart from "+", and $N=+$ and $M=-$, and using the fact that both F and G are null, one finds that

$$R_{\hat{N}\hat{M},L_1L_2}\Gamma^{L_1L_2}\eta_a = R_{+-,L_1L_2}\Gamma^{L_1L_2}\eta_a = 0 \ . \tag{5.9}$$

Since the isotropy group of 14 linearly independent spinors in $Spin(9,1)$ is $\{1\}$, one concludes that

$$R_{\hat{N}\hat{M},L_1L_2} = R_{+-,L_1L_2} = 0 \ , \tag{5.10}$$

and so the only non-vanishing components of the Riemann tensor are $R_{+i,+j}$.

To proceed further, it is useful to define

$$\begin{aligned}(T^2_{NM})_{L_1L_2} &= R_{NM,L_1L_2} - \frac{1}{4}G_{NM}{}^{Q}G^*_{L_1L_2Q} - \frac{1}{3}F_{[N|L_1}{}^{Q_1Q_2Q_3}F_{|M]L_2Q_1Q_2Q_3} \ , \\ (T^4_{NM})_{L_1L_2L_3L_4} &= 2F_{NM[L_1}{}^{Q_1Q_2}F_{L_2L_3L_4]Q_1Q_2} \ . \end{aligned} \tag{5.11}$$

In which case, (5.8) can be rewritten as

$$\Big(\frac{1}{2}(T^2_{NM})_{L_1L_2}\Gamma^{L_1L_2} + \frac{1}{24}(T^4_{NM})_{L_1L_2L_3L_4}\Gamma^{L_1L_2L_3L_4}\Big)\eta_a = 0 \ . \tag{5.12}$$

As the only nonzero components of T^2 and T^4 are $(T^2_{+i})_{+j}$ and $(T^2_{+i})_{+\ell_1\ell_2\ell_3}$, respectively, the only non-identically vanishing components of the above equation are

$$\Big((T^2_{+i})_{+j}\Gamma^j + \frac{1}{6}(T^4_{+i})_{+\ell_1\ell_2\ell_3}\Gamma^{\ell_1\ell_2\ell_3}\Big)\Gamma_-\eta_a = 0 \ , \tag{5.13}$$

or equivalently

$$\Big((T^2_{+i})_{+j}\Gamma^j + \frac{1}{6}(T^4_{+i})_{+\ell_1\ell_2\ell_3}\Gamma^{\ell_1\ell_2\ell_3}\Big)\chi_a = 0 \ , \tag{5.14}$$

where $\chi_a \in \{e_{125}, e_{135}, e_{145}, e_{235}, e_{245}, e_{345}\}$. It is straightforward to analyse these conditions, and one finds that

$$T^2 = T^4 = 0 \ . \tag{5.15}$$

In particular, $T^4=0$ implies that

$$\Psi_{\alpha}{}^{\nu}\Psi_{\nu\bar{\beta}} + \frac{1}{4}\delta_{\alpha\bar{\beta}}\Psi^{\nu\bar{\sigma}}\Psi_{\nu\bar{\sigma}} = 0 \ , \tag{5.16}$$

where the holomorphic indices $\alpha,\beta,\nu,\mu = 1,2,3,4$ are taken with respect to ω. The condition $T^2=0$ expresses the Riemann tensor of the spacetime in terms of the fluxes and we shall return to it later.

It remains to solve the integrability condition (5.7). This is done in appendix C to find that

$$\nabla F = 0 \tag{5.17}$$

ie F is also covariantly constant.

Returning to the condition $T^2 = 0$, since both F and G are covariantly constant, one concludes that

$$\nabla R = 0 \tag{5.18}$$

ie the spacetime is a Lorentzian symmetric space. These have been classified in [17]. Since in addition the spacetime admits a ∇-parallel null vector field and the only non-vanishing components of the curvature are

$$R_{+i,+j} = \big(2f^2 + \frac{1}{2}f^2\Psi_{kl}\Psi^{kl}\big)\delta_{ij} \ , \tag{5.19}$$

the spacetime is a *plane wave* or equivalently a Cahen-Wallach space. The above components of the Riemann tensor determine the wave profile.

To find the background explicitly, since the fluxes and the Riemann curvature are covariantly constant, one can follow the analysis of [1] for the maximally supersymmetric plane wave. In particular, one can determine the fluxes at the origin of the symmetric space. Then they can be defined everywhere on spacetime by acting with the transitive group. Indeed, the expression for the spacetime geometry can be simplified somewhat by solving (5.16). Since Ψ is (1,1) and ω-traceless, it lies up to a $SU(4)$-transformation on the maximal torus of $\mathfrak{su}(4)$. Using this and (5.16), one finds that, without loss of generality, Ψ can be written as

$$\Psi = -h(e^1\wedge e^6 + e^2\wedge e^7 - e^3\wedge e^8 - e^4\wedge e^9) \ , \tag{5.20}$$

where h is a real constant. Adapting coordinates to the null vector field $fe^+ = dv$ and putting the plane wave in Brinkman coordinates, the solution can be written as

$$\begin{aligned}
ds^2 &= 2dv[du - (\ell^2 + 2h^2)\delta_{ij}x^ix^jdv] + \delta_{ij}dx^idx^j \\
G &= -2\sqrt{2}i\,\ell\, e^{i\phi}dv\wedge(dx^1\wedge dx^6 + dx^2\wedge dx^7 + dx^3\wedge dx^8 + dx^4\wedge dx^9) \ , \\
F &= 2hdv\wedge(dx^1\wedge dx^2\wedge dx^6\wedge dx^7 - dx^3\wedge dx^4\wedge dx^8\wedge dx^9) \ ,
\end{aligned} \tag{5.21}$$

where we have re-instated a constant parameter ℓ using a coordinate transformation $v\to\ell^{-1}v$, $u\to\ell u$ and redefining the parameter as $h\to\ell h$. In the form given in (5.21) the solution depends on two parameters (ℓ, h) though one of them can be removed using a coordinate transformation provided that $\ell, h\neq 0$. However, the form given in (5.21) allows us to also consider the limits in which these parameters vanish. If either $h = 0$ or $\ell = 0$, the solution corresponds to either the heterotic solution of [13], G real, which preserves 14 supersymmetries or to the maximally supersymmetric plane wave solution of [3] respectively. If both F and G are non-vanishing, the solution preserves strictly 28 supersymmetries in IIB and it has been found in [7].

5.2 Solutions with $G = 0$

These backgrounds are a special case of the $SU(4)\ltimes\mathbb{R}^8$ solutions with $G = 0$ investigated in the previous section. So they are all locally maximally supersymmetric.

6 G_2-invariant normal

For solutions with a G_2-invariant normal ν_1, we take, without loss of generality,

$$\nu_1 = e_5 + e_{12345} \pm i(e_1 + e_{234}) \ . \tag{6.1}$$

In particular, ν_1 is invariant under G_2 transformations generated by

$$R^p\big(\Gamma_{1p} - \Gamma_{\bar{1}p} + \frac{1}{2}\epsilon_p{}^{\bar{q}_1\bar{q}_2}\Gamma_{\bar{q}_1\bar{q}_2}\big) + R^{\bar{p}}\big(\Gamma_{\bar{1}\bar{p}} - \Gamma_{1\bar{p}} + \frac{1}{2}\epsilon_{p}{}^{q_1q_2}\Gamma_{q_1q_2}\big) \ , \quad L^{p\bar{q}}\Gamma_{p\bar{q}} \ , \tag{6.2}$$

written in a manifestly $SU(3) \subset G_2$ covariant notation as in [10, 11, 12], where L is traceless $L^p{}_p = 0$ and $p, q = 2, 3, 4$. Therefore, L generates $SU(3)$ transformations in the $2, 3, 4$ directions.

To choose the second normal ν_2, consider the most general spinor linearly independent from ν_1,

$$\nu_2 = -x(e_5 - e_{12345}) - u^\alpha e_\alpha - \frac{1}{2}v^{\alpha\beta}e_{\alpha\beta5} - \frac{1}{6}w_\alpha\epsilon^{\alpha\beta_1\beta_2\beta_3}e_{\beta_1\beta_2\beta_3} \ , \tag{6.3}$$

where $\alpha = 1, 2, 3, 4$ and similarly for the rest of the indices.

By applying a $SU(3)$ transformation in the directions $2, 3, 4$ we can, without loss of generality, set $w_3 = w_4 = 0$, and then apply a $SU(2)$ transformation in the $3, 4$ directions to set $u^4 = 0$. Hence, we can choose

$$\begin{aligned}\nu_2 &= -x(e_5 - e_{12345}) - u^1e_1 - u^2e_2 - u^3e_3 - v^{12}e_{125} - v^{13}e_{135} - v^{14}e_{145} \\ &- v^{23}e_{235} - v^{24}e_{245} - v^{34}e_{345} - w_1e_{234} + w_2e_{134} \ .\end{aligned} \tag{6.4}$$

6.1 Solutions with $G \neq 0$

Given ν_1 and ν_2, we can compute G using (2.8). Then observe that the basis (η_a) in the space of Killing spinors includes

$$\{e_{35}, e_{45}, e_{1235}\} \ . \tag{6.5}$$

Evaluating the integrability condition (3.4) on these basis elements using CAC, one finds a number of relations, including

$$v^{23} = v^{24} = v^{14} = 0 \ . \tag{6.6}$$

This simplifies ν_2, and then the basis (η_a) of the Killing spinors includes the elements

$$\{e_{35}, e_{45}, e_{1235}, e_{14}, e_{13}, e_{23}\} \ . \tag{6.7}$$

Applying (3.4) to the above basis elements, one finds that

$$x = \mp iu^1 \ , \quad w_1 = -u^1 \ , \tag{6.8}$$

ie if one assumes, for example, that $x \neq \mp iu^1$, then (3.4) implies that $x = u^1 = 0$, and similarly for $w_1 = -u^1$. In addition, one obtains the conditions

$$w_2 = \mp iv^{34}, \quad u^3 = v^{13} = 0, \quad u^2 = \pm iv^{12} \ . \tag{6.9}$$

Next, applying (3.4) to the basis elements

$$\{e_{24}, e_{1245}, e_{34} \pm ie_{1345}, e_{12} \pm ie_{25}, 1 - e_{1234} \pm i(e_{15} - e_{2345}), 1 + e_{1234} \pm i(e_{2345} + e_{15})\} , \quad (6.10)$$

one finds that

$$v^{34} = u^1 = v^{12} = 0 \ . \quad (6.11)$$

Combining all the conditions implied by (3.4) on the components of ν_2 together, one concludes that $\nu_2 = 0$. This in turn gives $G = 0$ which is a contradiction. Thus there are no backgrounds with $G \neq 0$ in the G_2 case which preserve 28 supersymmetries.

6.2 Solutions with $G = 0$

Consider first the case for which $w_2 \neq 0$ in (6.4). By applying gauge transformations to ν_1, ν_2 generated by $R^1\Gamma_{-1} + R^{\bar{1}}\Gamma_{-\bar{1}}$ and $R^3\Gamma_{-3} + R^{\bar{3}}\Gamma_{-\bar{3}}$, one can eliminate the e_{345} and the e_{145} terms from ν_2. However, as these gauge transformations are *not* in G_2, the form of ν_1 is not left invariant under their action. Nevertheless, the simplification to ν_2 produced assists the computation. After these transformations, the two normals become

$$\begin{aligned} \nu_1 &= \alpha e_5 + \beta e_{12345} + \mu e_{135} + \nu e_{245} \pm i(e_1 + e_{234}) \\ \nu_2 &= \rho e_5 + \sigma e_{12345} - u^1 e_1 - u^2 e_2 - u^3 e_3 \\ &- s^{12} e_{125} - s^{13} e_{135} - s^{23} e_{235} - s^{24} e_{245} - w_1 e_{234} + w_2 e_{134} \ . \end{aligned} \quad (6.12)$$

To proceed, since $G = 0$, the only integrability condition that remains to be satisfied is

$$\mathcal{S}\eta_a \equiv (\frac{1}{2}T^2 + \frac{1}{24}T^4)\eta_a = 0 \ , \quad (6.13)$$

and it is given in detail in (4.23). Evaluating this integrability condition on the basis elements

$$\begin{aligned} &\{e_{35}, e_{45}, e_{1235}, e_{23}, e_{12}, w_2 e_{1345} + u^2 e_{25}, w_2 e_{1245} - u^3 e_{25}, w_2 e_{34} + s^{12} e_{25}, \\ &w_2 e_{14} + s^{23} e_{25}, w_2(e_{15} - e_{2345}) + (u^1 - w_1)e_{25}, w_2 1 \pm i\beta w_2 e_{15} + (\sigma \mp i\beta w_1)e_{25}, \\ &w_2 e_{24} \pm i\mu w_2 e_{15} + (-s^{13} \mp i\mu w_1)e_{25}, w_2 e_{13} \pm i\nu w_2 e_{15} + (-s^{24} \mp i\nu w_1)e_{25}\} \end{aligned} \quad (6.14)$$

one finds after some CAC that all components of T^2 and T^4 vanish. Therefore all these solutions are locally maximally supersymmetric.

Next, consider the case for which $w_2 = 0$. Then by making a $SU(2)$ rotation in the $2,3$ directions, one can set, without loss of generality, $u^3 = 0$ also. Suppose first that that $u^2 \neq 0$. By applying a gauge transformation generated by $R^1\Gamma_{-1} + R^{\bar{1}}\Gamma_{-\bar{1}}$ to ν_1, ν_2, one can eliminate the e_{125} term from ν_2. Again the form of ν_1 is altered, because this transformation is not in G_2. Then apply a $SU(2)$ rotation in the $3,4$ directions to

eliminate the e_{245} term from ν_2. We therefore obtain

$$\begin{aligned}\nu_1 &= \alpha e_5 + \beta e_{12345} \pm i(e_1 + e_{234}) \\ \nu_2 &= \rho e_5 + \sigma e_{12345} - u^1 e_1 - u^2 e_2 \\ &- s^{13} e_{135} - s^{14} e_{145} - s^{23} e_{235} - s^{34} e_{345} - w_1 e_{234} \ .\end{aligned} \tag{6.15}$$

To proceed, evaluate the $\mathcal{S}\eta_a = 0$ integrability condition on the basis elements

$$\begin{aligned}&\{e_{35}, e_{45}, e_{1235}, e_{25}, e_{1245}, e_{34}, e_{13}, u^2 e_{24} + s^{13} e_{1345}, u^2 e_{23} - s^{14} e_{1345} \\ &u^2 e_{14} - s^{23} e_{1345}, u^2 e_{12} - s^{34} e_{1345}, u^2(e_{2345} - e_{15}) + (u^1 - w_1) e_{1345}, \\ &u^2 1 \pm i\beta u^2 e_{15} + (-\sigma \pm i\beta w_1) e_{1345} \ , \}\end{aligned} \tag{6.16}$$

to find that all components of T^2 and T^4 vanish. Thus again these solutions are locally maximally supersymmetric.

Next consider the case for which $w_2 = u^2 = u^3 = 0$. One can then apply a $SU(3)$ transformation to set, without loss of generality $v^{13} = v^{14} = 0$, followed by a $SU(2)$ transformation in the $3, 4$ directions to set $v^{24} = 0$. After doing this, we have

$$\begin{aligned}\nu_1 &= e_5 + e_{12345} \pm i(e_1 + e_{234}) \\ \nu_2 &= -x(e_5 - e_{12345}) - u^1 e_1 - v^{12} e_{125} - v^{23} e_{235} - v^{34} e_{345} - w_1 e_{234} \ .\end{aligned} \tag{6.17}$$

Suppose that $v^{12} \neq 0$. Then apply the integrability condition $\mathcal{S}\eta_a = 0$ to the basis elements

$$\begin{aligned}&\{e_{35}, e_{45}, e_{1235}, e_{1345}, e_{1245}, e_{24}, e_{23}, e_{13}, e_{25}, v^{12} e_{14} - v^{23} e_{34}, v^{12} e_{12} - v^{34} e_{34}, \\ &v^{12}(e_{2345} - e_{15}) + (u^1 - w_1) e_{34}, v^{12} 1 \pm i v^{12} e_{15} + (-x \pm i w_1) e_{34}\} \ ,\end{aligned} \tag{6.18}$$

one finds that all components of T^2 and T^4 vanish, and so the solutions are again locally maximally supersymmetric.

Finally, consider the case for which $w_2 = u^2 = u^3 = v^{13} = v^{14} = v^{24} = v^{12} = 0$. By making a $SU(2)$ transformation in the $2, 4$ directions, one can also set, without loss of generality, $v^{23} = 0$. In such a case,

$$\begin{aligned}\nu_1 &= e_5 + e_{12345} \pm i(e_1 + e_{234}) \\ \nu_2 &= -x(e_5 - e_{12345}) - u^1 e_1 - v^{34} e_{345} - w_1 e_{234} \ .\end{aligned} \tag{6.19}$$

Next, note that for solutions preserving exactly $N = 28$ supersymmetries, there must be a basis (η_a) for the Killing spinors which contains the elements

$$\begin{aligned}\{&e_{35}, e_{45}, e_{1235}, e_{1345}, e_{1245}, e_{24}, e_{23}, e_{13}, e_{25}, e_{34}, e_{14}, \\ &z_1 1 + z_2 e_{1234} + z_3 e_{15} + z_4 e_{2345} + z_5 e_{12}\}\end{aligned} \tag{6.20}$$

where z_1, z_2, z_3, z_4 do not all vanish. Note that this is clearly true if $v^{34} \neq 0$. In the special case for which $v^{34} = 0$, e_{12} can be taken as a basis element, but for a $N = 28$

solution, two further basis elements must also be found, and such a solution therefore still has a basis containing the spinors in (6.20). On evaluating the integrability condition $\mathcal{S}\eta_a = 0$ on these basis elements, one finds that all components of $T^2 = T^4 = 0$. Thus again these solutions are locally maximally supersymmetric.

Therefore, we have shown that if one of the normals is G_2-invariant, then all solutions with $N = 28$ supersymmetries are locally maximally supersymmetric.

7 Discrete Quotients

We have demonstrated that all $N = 28$ supersymmetric IIB backgrounds are either locally isometric to that of [7] or to a maximally supersymmetric background. The possibility remains that some $N = 28$ backgrounds can be constructed as discrete quotients of maximally supersymmetric ones. Here, we shall prove that all discrete quotients of maximally supersymmetric backgrounds preserve less than 28 supersymmetries, $N < 28$. So there are no $N = 28$ backgrounds which arise as discrete quotients of maximally supersymmetric ones. To show this, we shall use the machinery developed in [18]. This has been applied both in M-theory [19] to prove that there are no $N = 31$ quotients of maximally supersymmetric backgrounds and in IIB supergravity [5] to demonstrate a similar result for backgrounds with $N > 28$ supersymmetries.

7.1 Discrete quotients of Minkowski space

This computation is similar to that we have performed in [5] to search for backgrounds with $N > 28$ supersymmetries, so we shall not give an extensive description of the analysis. To find discrete quotients of Minkowski space which preserve 28 supersymmetries, one has to find an element $\alpha \in SO(9,1)$ such that its lift $\hat{\alpha} \in Spin(9,1)$ preserves 28 spinors, ie it acts as the identity on a 28-dimensional subspace of the Weyl representation $\Delta_{\mathbf{16}}$ of $Spin(9,1)$. Up to a conjugation, there are two choices for the lift $\hat{\alpha}$. One choice is that $\hat{\alpha}$ can be written as

$$\hat{\alpha} = \exp[\frac{1}{2}(\theta_0\Gamma_{05} + \theta_1\Gamma_{16} + \theta_2\Gamma_{27} + \theta_3\Gamma_{38} + \theta_4\Gamma_{48}) + i\psi] \ , \tag{7.1}$$

where the additional angle ψ has been added because of the $Spin_c(9,1)$ nature of spinors of IIB supergravity. Decomposing $\Delta_{\mathbf{16}}$ as

$$\Delta_{\mathbf{16}} = \sum_{\sigma_0,\dots,\sigma_4=\pm1} W_{\sigma_0\dots\sigma_4} \tag{7.2}$$

using the projectors $\Gamma_i\Gamma_{i+5}$, $i = 0, \dots, 4$, the lifted element can be written as

$$\hat{\alpha}(\sigma_0, \dots, \sigma_4) = \exp[\frac{1}{2}(\sigma_0\theta_0 + i\sigma_1\theta_1 + i\sigma_2\theta_2 + i\sigma_3\theta_3 + i\sigma_4\theta_4) + i\psi] \tag{7.3}$$

where the chirality condition requires that $\sigma_0\sigma_1 \dots \sigma_4 = 1$.

For an element $\hat{\alpha}$ to preserve 28 supersymmetries, it has to act as an identity on a 28 dimensional subspace V of $\Delta_{\mathbf{16}}$. In particular, there are some $\sigma_0, \dots, \sigma_4$, such that

$$\hat{\alpha}(\sigma_0, \sigma_1, \dots, \sigma_4) = 1 \ . \tag{7.4}$$

Taking the complex conjugate, one concludes that $\theta_0 = 0$. So boosts do not preserve any supersymmetry as expected. Since $\theta_0 = 0$, $\hat{\alpha}$ is independent of σ_0. So in what follows we shall explicitly indicate the dependence of subspaces W and the map $\hat{\alpha}$ on only the rest of the signs.

To exclude the possibility that some spatial rotations preserve 28 supersymmetries, $\hat{\alpha}$ must not act as the identity on two $W_{\sigma_1\dots\sigma_4}$ subspaces. It is straightforward to observe that whatever the choice of non-invariant subspaces is, there is always a choice of signs such that

$$\hat{\alpha}(\sigma_1, \dots, \sigma_4) = \hat{\alpha}(-\sigma_1, \dots, -\sigma_4) = 1 \ . \tag{7.5}$$

This in particular implies that $\exp(2i\psi) = 1$. Using this, one can show that if for some signs $\hat{\alpha}(\sigma_1, \dots, \sigma_4) = 1$, then $\bar{\hat{\alpha}}(\sigma_1, \dots, \sigma_4) = \hat{\alpha}(-\sigma_1, \dots, -\sigma_4) = 1$. Therefore if the action on $W_{\sigma_1\dots\sigma_4}$ is trivial, so is the action on the conjugate module $W_{-\sigma_1\dots-\sigma_4}$. Thus in order to preserve precisely 28 supersymmetries, the two non-invariant subspaces should be chosen to be conjugate to each other.

To proceed since all choices of the signs are symmetric, without loss of generality, assume that

$$\hat{\alpha}(+1, +1, +1, +1) = \bar{\hat{\alpha}}(-1, -1, -1, -1) \neq 1 \ . \tag{7.6}$$

To preserve precisely 28 supersymmetries, for all other choices of signs $\hat{\alpha}$ must be the identity. In particular,

$$\hat{\alpha}(-1, +1, +1, +1) = \hat{\alpha}(1, -1, -1, +1) = 1 \ . \tag{7.7}$$

This implies that $\exp(i\theta_4) = 1$. This gives

$$\exp(i\theta_4)\hat{\alpha}(+1, +1, +1, -1) = \hat{\alpha}(1, 1, 1, 1) = 1 \tag{7.8}$$

which is a contradiction. Thus if one assumes that a 28-dimensional subspace of $\Delta_{\mathbf{16}}$ is invariant under some $\hat{\alpha}$, then all $\Delta_{\mathbf{16}}$ is invariant and so all supersymmetry is preserved. There are no such quotients with preserve 28 supersymmetries.

The another choice for $\hat{\alpha}$ is to take

$$\hat{\alpha} = \exp\{\frac{1}{2}[(\Gamma_0 + \Gamma_5)\Gamma_9 + \theta_1\Gamma_{16} + \theta_2\Gamma_{27} + \theta_3\Gamma_{38}]\} \ . \tag{7.9}$$

Decomposing $\Delta_{\mathbf{16}}$ using the projector Γ_{05}, it is easy to see that such quotients preserve at most 16 supersymmetries.

7.2 Discrete quotients of $AdS_5 \times S^5$

The isometry group of the $AdS_5 \times S^5$ background is $SO(4,2) \times SO(6)$. To find whether there is a discrete subgroup D of $SO(4,2) \times SO(6)$ such that $AdS_5 \times S^5/D$ preserves 28 supersymmetries, observe that the associated spin group $Spin(4,2) \times Spin(6)$ acts on $\Delta_{\mathbf{16}}$ as $\Delta^-_{Spin(4,2)} \times \Delta^-_{Spin(6)}$, where $\Delta^-_{Spin(4,2)}$ and $\Delta^-_{Spin(6)}$ are the chiral representations of $Spin(4,2)$ and $Spin(6)$, respectively.

It is a consequence of the tensor product structure of the representation of $Spin(4,2)\times Spin(6)$ on $\Delta_{\mathbf{16}}$ that the real dimension of an invariant subspace V of $\hat{\alpha}$ is

$$\mathrm{dim}V = 2nm \ , \quad 1 \leq n, m \leq 4 \ . \tag{7.10}$$

Since 28 cannot be written as a product in this way, there are no discrete quotients of $AdS_5 \times S^5$ which preserve 28 supersymmetries. In fact this argument implies that the largest number of supersymmetries, less than maximal, which can be preserved by a discrete $AdS_5 \times S^5$ quotient[3] is 24.

7.3 Discrete quotients of Maximally supersymmetric plane wave

To investigate the existence of discrete quotients of the maximally supersymmetric plane wave which preserve 28 supersymmetries, we shall follow closely the analysis in [5]. In particular, it has been shown that the invariance condition for $\hat{\alpha}$, $\hat{\alpha}\epsilon = \epsilon$, can be written as

$$\begin{aligned} e^A \epsilon_- &= \epsilon_- \ , \\ e^A(\epsilon_+ + \Gamma_+ \beta \epsilon_-) &= \epsilon_+ \ , \end{aligned} \tag{7.11}$$

where $\Gamma_+\epsilon_+ = 0$ is the usual light-cone projection. Moreover, one can show that

$$\hat{\alpha}(\sigma_1, \dots, \sigma_4)\epsilon_- = e^A \epsilon_- = \exp\big[\frac{i}{2}\sum_{i=1}^{4}\sigma_i\theta_i + i\psi\big]\epsilon_- \ , \quad \sigma_1\sigma_2\sigma_3\sigma_4 = -1 \ , \tag{7.12}$$

and

$$e^A \, \epsilon_+ = \exp\big[-2i\lambda v^- \sigma_1\sigma_2 + \frac{i}{2}\sum_{i=1}^{4}\sigma_i\theta_i + i\psi\big]\, \epsilon_+ \ , \quad \sigma_1\sigma_2\sigma_3\sigma_4 = 1 \ . \tag{7.13}$$

In particular, the part of A that depends on v^- acts with the identity on ϵ_-. Decomposing $\Delta_{\mathbf{16}} = V_- \oplus V_+$ using the lightcone projection, it has been shown in [5] that there is no quotient which preserves more than 28 supersymmetries.

To extended the above result to the $N = 28$ case, there are two possibilities. Either the discrete group action leaves invariant a 6-dimensional subspace in V_- and acts as the identity on V_+ or vice versa. Consider first the former possibility. As has already been indicated in (7.12) and (7.13), we have decomposed both V_- and V_+ into eight 1-dimensional complex subspaces $W_{\sigma_1\dots\sigma_4}$ and $Z_{\sigma_1\dots\sigma_4}$, respectively, labeled by the eight independent choices of signs σ. It is easy to see that whatever the choice of the 6-dimensional invariant subspace of V_- is, one can show that $e^{2i\psi} = 1$. Using this, one can also show that if a subspace $W_{\sigma_1\dots\sigma_4}$ is invariant so is the subspace $W_{\bar{\sigma}_1\dots\bar{\sigma}_4}$ with $\bar{\sigma}_i = -\sigma_i$. Thus for e^A to preserve precisely a 6-dimensional subspace of V_-, the non-invariant 2-dimensional complex subspace of V_- must be as $W_{\sigma_1\dots\sigma_4} \oplus W_{\bar{\sigma}_1\dots\bar{\sigma}_4}$ for some choice of σ_i. Since the choice of signs is symmetric, without loss of generality, one can

[3]The addition of the angle ψ due to the $Spin_c$ nature of IIB spinors does not affect this argument.

choose $W_{+1,+1,+1,-1} \oplus W_{-1,-1,-1,+1}$ as the non-invariant subspace. Solving the condition $e^A = 1$ for the remaining choices of signs, one finds that

$$\theta_1 = \theta \ , \quad \theta_2 = \theta + 2\pi n_2 \ , \quad \theta_3 = \theta + 2\pi n_3 \ , \quad \theta_4 = -\theta + 2\pi n_4 \ , \tag{7.14}$$

and

$$\pi(n_2 + n_3 + n_4) + \psi \in 2\pi\mathbb{Z} \ , \quad n_2, n_3, n_4 \in \mathbb{Z} \ , \tag{7.15}$$

where θ is an arbitrary angle and $\psi = n\pi$, $n \in \mathbb{Z}$. So there are transformations which preserve a six dimensional subspace I of V_-.

To preserve precisely 28 supersymmetries, all V_+ must be invariant under the action of the discrete group. For this it is necessary that $I \subset \mathrm{Ker}\,\beta$ and that e^A acts as the identity on V_+. It is always possible to choose the group action to satisfy the first condition. So let as focus on the second. In particular, the invariance of the subspaces $Z_{+1,+1,+1,+1}$ and $Z_{-1,-1,-1,-1}$ imply that

$$e^{-2i\lambda v^- + i\theta} = e^{-2i\lambda v^- - i\theta} = 1 \ . \tag{7.16}$$

This in turn gives

$$2\lambda v^- = n_0\pi \ , \quad \theta = n_1\pi \ , \quad n_0 + n_1 \in 2\mathbb{Z} \ , \quad n_0, n_1 \in \mathbb{Z} \ . \tag{7.17}$$

However now notice that for this choice of θ, $W_{+1,+1,+1,-1}$ and so $W_{-1,-1,-1,+1}$ are also invariant, ie all V_- is preserved. In such a case, the only option for preserving 28 supersymmetries is that $\dim_{\mathbb{C}}\mathrm{Ker}\,\beta = 6$. However, it is easy to see that the dimension of the kernel of β is either 4 or 8. So such quotients cannot preserve strictly 28 supersymmetries.

Next suppose that the discrete symmetry preserves all V_-. In such a case, the angles θ_i are given as in (7.14) and (7.15), and $\theta = n_1\pi$, $n_1 \in \mathbb{Z}$. For the quotient to preserve precisely 28 supersymmetries, one should choose the discrete subgroup that $\dim_{\mathbb{C}}\mathrm{Ker}\beta = 8$. As we have already mentioned there is always such a choice. We require that e^A leaves invariant a complex 6-dimensional subspace of V_+. In particular, note that

$$\exp\left[i\sum_{i=1}^{4}\sigma_i\theta_i\right] = 1 \tag{7.18}$$

where $\sigma_1\sigma_2\sigma_3\sigma_4 = 1$, and θ_i are constrained as above. It follows that if $Z_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}$ is an invariant subspace, then so is $Z_{-\sigma_1,-\sigma_2,-\sigma_3,-\sigma_4}$. Thus for e^A to preserve precisely a 6-dimensional subspace of V_+, the non-invariant 2-dimensional complex subspace of V_+ must be as $Z_{\sigma_1\dots\sigma_4} \oplus Z_{-\sigma_1\dots-\sigma_4}$ for some choice of σ_i. Since all choices are symmetric, take as the non-invariant subspace $Z_{+1,+1,+1,+1} \oplus Z_{-1,-1,-1,-1}$. Requiring that e^A leave invariant the 6-dimensional subspace complementary to $Z_{+1,+1,+1,+1} \oplus Z_{-1,-1,-1,-1}$ imposes the condition

$$e^{-2i\lambda v^- + i\pi n_1} = 1 \ . \tag{7.19}$$

However, this condition also implies that $Z_{+1,+1,+1,+1}$ is an invariant subspace, and so all V_+ is invariant. Thus all the supersymmetry is preserved, and there are no quotients that preserve strictly 28 supersymmetries.

8 Strings in the plane wave background

8.1 Geometry of plane wave

As we have already mentioned, the plane wave solution (5.21) is the superposition of two other plane wave solutions, those of the maximally supersymmetric plane wave of [3] and the heterotic plane wave preserving 14 supersymmetries[4]. These two solutions are also recovered in the limits of (5.21) for which the parameters (ℓ, h) vanish.

We have shown that (5.21) is a Lorentzian symmetric space and the form fluxes are parallel. In fact the spacetime is a Lorentzian Lie group because the wave profile is negative definite. The isometries of the metric are precisely those of the maximally supersymmetric plane wave which have been investigated in [3]. In particular, the algebra of Killing vector fields is $\mathfrak{so}(8) \ltimes \mathfrak{h}(8)$, where $\mathfrak{h}(8)$ is the Heisenberg Lie algebra extended by an outer $\mathfrak{u}(1)$ automorphism which rotates the 8 positions to the 8 momenta and commutes with the central element. However the fluxes are not invariant under the whole group of isometries. The 5-form flux, as is well known, breaks this group to $(\mathfrak{so}(4) \oplus \mathfrak{so}(4)) \ltimes \mathfrak{h}(8)$. The additional 3-form flux of (5.21) breaks the isometry group further to $(\mathfrak{u}(2) \oplus \mathfrak{u}(2)) \ltimes \mathfrak{h}(8)$ which is the symmetry group of the background. The $\mathfrak{u}(2) \oplus \mathfrak{u}(2)$ is identified as the subalgebra of $\mathfrak{so}(4) \oplus \mathfrak{so}(4)$ which in addition preserves a complex structure on the transverse directions to the lightcone. Moreover observe that in the limit that the 5-form flux vanishes, the symmetry group of the background enhances to $\mathfrak{u}(4) \ltimes \mathfrak{h}(8)$.

8.2 String propagation

The worldvolume dynamics of a string in the (5.21) background is described by a Green-Schwarz action. To quantize string theory, one has to gauge fix the kappa symmetry and rewrite the theory in terms of worldvolume fermions. In this case, this procedure is considerably simplified because the background is a plane wave and it admits a natural lightcone gauge. In particular, the resulting action is always quadratic in the worldvolume fermions [21]. We shall not carry out this procedure in detail. Instead, we shall use the close relation that this theory has with the maximally supersymmetric plane wave and argue that the bosonic part of the string action is that of a string on a plane wave group manifold

$$\begin{aligned} ds^2 &= 2dudv - (\ell^2 + 4h^2)x^2 du^2 + dx^2 \ , \\ G &= -2\,\ell\, du \wedge \omega \ , \qquad \omega = (dx^1 \wedge dx^6 + dx^2 \wedge dx^7 + dx^3 \wedge dx^8 + dx^4 \wedge dx^9) \end{aligned} \tag{8.1}$$

where we re-scale ℓ to $\ell/\sqrt{2}$. (The normalization of the fluxes is consistent with that of [22].) In particular, the 5-form flux does not contribute in the bosonic part of the action apart from the h^2 contribution in the metric. However it is expected to contribute in the fermion couplings.

The quantization of strings on a (8.1) background is a special case of the models investigated in [22], see also eg [23, 24, 25]. Here we shall carry out some of the steps in

[4]See [13] and [20] for a general discussion of heterotic solutions with more than 8 supersymmetries.

the analysis of [22] to identify the lightcone string Hamiltonian. We shall show that this Hamiltonian is a linear superposition of infinite many Harmonic oscillator Hamiltonians. To find the frequencies of these Harmonic oscillators, we first use a frequency based ansatz to solve the classical string equations. In particular, one finds that the classical frequencies $\tilde{\omega}$ satisfy the equation

$$\det\big((\tilde{\omega}^2 - \ell^2 - 4h^2 - 4n^2)\delta_{ij} - 4in\ell\omega_{ij}\big) = 0 \ , \quad n \in \mathbb{Z} \tag{8.2}$$

which gives

$$[(\tilde{\omega}^2 - \ell^2 - 4h^2 - 4n^2)^2 - 16n^2\ell^2]^4 = 0 \ . \tag{8.3}$$

The center of mass mode, $n = 0$, has a single frequency

$$(\tilde{\omega}^{(0)})^2 = \ell^2 + 4h^2 \ . \tag{8.4}$$

For the other modes one has

$$(\tilde{\omega}^{(n)}_{\pm})^2 = \pm 4n\ell + \ell^2 + 4h^2 + 4n^2 \ . \tag{8.5}$$

Observe that all frequency squares are positive for $h > 0$.

It has been shown in [22] that the classical frequencies of the string after quantization are identified with the quantum frequencies of the lightcone string Hamiltonian. Moreover $\tilde{\omega}^{(n)}_{\pm} = \tilde{\omega}^{(-n)}_{\mp}$ and so the n and $-n$ modes pair. The lightcone Hamiltonian of the string can be written as

$$H = \sum_{n \geq 0} H^{(n)} \tag{8.6}$$

where $H^{(n)}$ is the sum of appropriate Harmonic oscillator Hamiltonians. In particular, one finds that

$$H^{(0)} = \sum_{j=1}^{8} \tilde{\omega}^{(0)}(\mathcal{N}_j + \frac{1}{2}) \ , \quad \mathcal{N}_j = \mathrm{a}^\dagger_j\, \mathrm{a}_j \ , \tag{8.7}$$

and

$$H^{(n)} = \sum_{i=1}^{8} \tilde{\omega}^{(n)}_{+}(\mathcal{N}_{+}{}^{(n)}_{i} + \frac{1}{2}) + \sum_{j=1}^{8} \tilde{\omega}^{(n)}_{-}(\mathcal{N}_{-}{}^{(n)}_{j} + \frac{1}{2}) \ , \quad \mathcal{N}_{\pm}{}^{(n)}_{j} = \mathrm{a}_{\pm}{}^{(n)\dagger}_{j}\, \mathrm{a}_{\pm}{}^{(n)}_{j} \ . \tag{8.8}$$

The operators $\mathrm{a}^\dagger_j$, $\mathrm{a}_{\pm}{}^{(n)\dagger}_{j}$ and a_j, $\mathrm{a}_{\pm}{}^{(n)}{}_{j}$ are creation and annihilation operators, respectively, canonically normalized as those of a Harmonic oscillator. So we have shown that the center of mass Hamiltonian comprises of 8 Harmonic oscillators with the same frequency and each oscillator mode $n > 0$ comprises of 8 Harmonic oscillators with frequency $\tilde{\omega}^{(n)}_{+}$ and 8 Harmonic oscillators with frequency $\tilde{\omega}^{(n)}_{-}$.

9 Outlook

We have shown that the IIB supersymmetric backgrounds with strictly 28 supersymmetries are locally isometric to the solution of [7]. Combining this with the classification of the maximally supersymmetric backgrounds of IIB supergravity in [1] and the results of [4, 5] gives a classification of all supersymmetric backgrounds of IIB supergravity with more than 27 supersymmetries, $N > 27$. The conjecture of [26] is consistent with our result. Moreover, we have demonstrated that IIB backgrounds with only 5-form flux that admit more than 26 supersymmetries, $N > 26$, are maximally supersymmetric.

It is not known whether there are IIB solutions which preserve 25, 26 or 27 supersymmetries. However, it is known that there is a plane wave solution which preserves 24 supersymmetries [7]. This is again a superposition of the maximally supersymmetric plane wave with a plane wave solution of the heterotic string which preserves 12 supersymmetries. Since there is a unique heterotic solution which preserves 12 supersymmetries and there are no solutions which preserve 13 supersymmetries, it is tempting to propose that the IIB $N = 24$ solution is unique and there are no IIB solutions with 25, 26 and 27 supersymmetries. However, there is no firm evidence for this apart from the analogy with the plane-wave solutions of the heterotic string.

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Appendix A The normals to the Killing spinors

A.1 $SU(4) \ltimes \mathbb{R}^8$

A.1.1 Second normal

A representative for the first $SU(4) \ltimes \mathbb{R}^8$-invariant normal spinor [4, 5] is

$$\nu_1 = -pe_5 - qe_{12345} \ , \tag{A.1}$$

where $|p| \neq |q|$. The infinitesimal generators of the $SU(4) \ltimes \mathbb{R}^8$ isotropy group are

$$L^{\alpha\bar{\beta}}\Gamma_{\alpha\bar{\beta}} \ , \quad R^{\alpha}\Gamma_{-\alpha} + R^{\bar{\alpha}}\Gamma_{-\bar{\alpha}} \ , \quad \alpha, \beta = 1, 2, 3, 4 \ , \tag{A.2}$$

where $L \in \mathfrak{su}(4)$, ie $L^{\alpha\bar{\beta}}\delta_{\alpha\bar{\beta}} = 0$, and $R^{\alpha} = (R^{\bar{\alpha}})^*$.

A basis in the anti-chiral $Spin(9, 1)$ representation $\Delta^-_{\mathbf{16}}$ can be chosen as

$$e_5 \ , \quad e_{12345} \ , \quad e_{\mu} \quad e_{\mu\nu\rho} \ , \quad e_{\mu\nu 5} \quad . \tag{A.3}$$

$\Delta^-_{\mathbf{16}}$ is decomposed under $SU(4)$ as $\mathbf{16} = \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{4} \oplus \bar{\mathbf{4}} \oplus \mathbf{6}$. To find a representative for

the second normal, consider the following formulae,

$$
\begin{aligned}
\big(R^\alpha \Gamma_{-\alpha} + R^{\bar\alpha}\Gamma_{-\bar\alpha}\big)e_\mu &= 2R^\alpha e_{\alpha\mu 5} + 2R_\mu e_5 \ , \\
\big(R^\alpha \Gamma_{-\alpha} + R^{\bar\alpha}\Gamma_{-\bar\alpha}\big)e_{\mu\nu 5} &= 0 \ , \\
\big(R^\alpha \Gamma_{-\alpha} + R^{\bar\alpha}\Gamma_{-\bar\alpha}\big)e_{\mu\nu\rho} &= 2R^\alpha \epsilon_{\alpha\mu\nu\rho} e_{12345} + 6R_{[\mu} e_{\nu\rho]5} \ ,
\end{aligned} \tag{A.4}
$$

and

$$
\begin{aligned}
L^{\alpha\bar\sigma}\Gamma_{\alpha\bar\sigma} e_\mu &= 2L^\alpha{}_\mu e_\alpha \\
L^{\alpha\bar\sigma}\Gamma_{\alpha\bar\sigma} e_{\mu\nu 5} &= -4L^\alpha{}_{[\mu} e_{\nu]\alpha 5} \ , \\
L^{\alpha\bar\sigma}\Gamma_{\alpha\bar\sigma}\epsilon^{\gamma\beta_1\beta_2\beta_3} e_{\beta_1\beta_2\beta_3} &= -2L^\gamma{}_\rho \epsilon^{\rho\beta_1\beta_2\beta_3} e_{\beta_1\beta_2\beta_3} \ .
\end{aligned} \tag{A.5}
$$

It is also useful to consider the gauge transformations generated by Γ_{+-} and $i\delta^{\alpha\bar\beta}\Gamma_{\alpha\bar\beta}$ which act on spinors as

$$
\begin{aligned}
e^{f_1\Gamma_{+-} + i f_2 \delta^{\alpha\bar\beta}\Gamma_{\alpha\bar\beta}} e_5 &= e^{-f_1 - 4if_2} e_5 \ , \\
e^{f_1\Gamma_{+-} + i f_2 \delta^{\alpha\bar\beta}\Gamma_{\alpha\bar\beta}} e_{12345} &= e^{-f_1 + 4if_2} e_{12345} \ , \\
e^{f_1\Gamma_{+-} + i f_2 \delta^{\alpha\bar\beta}\Gamma_{\alpha\bar\beta}} e_\mu &= e^{f_1 - 2if_2} e_\mu \ , \\
e^{f_1\Gamma_{+-} + i f_2 \delta^{\alpha\bar\beta}\Gamma_{\alpha\bar\beta}} e_{\mu\nu 5} &= e^{-f_1} e_{\mu\nu 5} \ , \\
e^{f_1\Gamma_{+-} + i f_2 \delta^{\alpha\bar\beta}\Gamma_{\alpha\bar\beta}} e_{\mu\nu\rho} &= e^{f_1 + 2if_2} e_{\mu\nu\rho} \ .
\end{aligned} \tag{A.6}
$$

Although these transformations do not leave e_5 and e_{12345} invariant, they do leave the plane spanned of e_5 and e_{12345} invariant. So they are generators of the Σ group [20].

Now suppose that

$$
\nu_2 = -Xe_5 - Ye_{12345} - u^\alpha e_\alpha - \frac{1}{2} v^{\alpha\beta} e_{\alpha\beta 5} - \frac{1}{6} w_\alpha \epsilon^{\alpha\beta_1\beta_2\beta_3} e_{\beta_1\beta_2\beta_3} \ , \tag{A.7}
$$

is the second normal spinor.

Using a $SU(4)$ transformation, we can without loss of generality set $w_2 = w_3 = w_4 = 0$, with $w_1 = w$. The isotropy group is $SU(3)$. By applying a $SU(3)$ transformation in the $2,3,4$ directions, one can without loss of generality also set $u^3 = u^4 = 0$. So we have

$$
\nu_2 = -Xe_5 - Ye_{12345} - u^1 e_1 - u^2 e_2 - \frac{1}{2} v^{\alpha\beta} e_{\alpha\beta 5} - we_{234} \ . \tag{A.8}
$$

Next apply a $\mathbb{R}^8$ gauge transformation generated by $R^\alpha\Gamma_{-\alpha} + R^{\bar\alpha}\Gamma_{-\bar\alpha}$, which maps

$$
\begin{aligned}
\nu_2 \to \nu_2' &= \big(-X - 2R_1u^1 - 2R_2u^2\big)e_5 + \big(-Y - 2wR^1\big)e_{12345} - u^1e_1 - u^2e_2 - we_{234} \\
&+ \big(-v^{1p} + 2u^1R^p - 2u^pR^1\big)e_{1p5} + \big(-\frac{1}{2}v^{p_1p_2} + 2u^{[p_1}R^{p_2]} - w\epsilon^{qp_1p_2}R_q\big)e_{p_1p_25} \ ,
\end{aligned} \tag{A.9}
$$

for $p, q = 2, 3, 4$.

First consider the case for which $|u^1|^2+|u^2|^2 \neq 0$. Then one can choose R^1, R^2, R^3, R^4 in order to eliminate the e_5 and e_{1p5} terms ($p = 2, 3, 4$), giving

$$\nu_2 = -y e_{12345} - u^1 e_1 - u^2 e_2 - w e_{234} - \frac{1}{2}(v')^{p_1 p_2} e_{p_1 p_2 5} \ . \tag{A.10}$$

In addition by applying a $SU(2)$ transformation in the $3, 4$ directions, one can remove the e_{245} term, to leave

$$\nu_2 = -y e_{12345} - u^1 e_1 - u^2 e_2 - w e_{234} - c_3 e_{235} - c_4 e_{345} \ . \tag{A.11}$$

Next consider the case for which $u^1 = u^2 = 0$. In this case applying the $\mathbb{R}^8$ transformation gives

$$\nu_2' = -X e_5 + \big(-Y - 2wR^1\big) e_{12345} - w e_{234} - v^{1p} e_{1p5} + \big(-\frac{1}{2} v^{p_1 p_2} - w \epsilon^{q p_1 p_2} R_q\big) e_{p_1 p_2 5} \ . \tag{A.12}$$

If $w \neq 0$, then one can choose R^1, R^2, R^3, R^4 in order to eliminate the $e_{12345}, e_{235}, e_{245}, e_{345}$ terms, giving

$$\nu_2 = -X e_5 - w e_{234} - v^{1p} e_{1p5} \ . \tag{A.13}$$

Then, applying a $SU(3)$ transformation in the $2, 3, 4$ directions, the e_{125} and e_{135} terms can also be removed to give

$$\nu_2 = -x e_5 - w e_{234} - c_3 e_{145} \ . \tag{A.14}$$

However, note that this (A.14) is gauge equivalent to a special case of (A.11). The gauge transformation used to relate the two ν_2 is $\Gamma_{1234} = e^{\frac{\pi}{2}(\Gamma_{12}+\Gamma_{34})}$ (here the indices are in the *real* basis). Furthermore, this gauge transformation also preserves the span of e_5 and e_{12345}. Hence we can discard the case when $w \neq 0$.

The remaining case therefore has $u^1 = u^2 = w = 0$. Then

$$\nu_2 = -X e_5 - Y e_{12345} - \frac{1}{2} v^{\alpha\beta} e_{\alpha\beta 5} \ . \tag{A.15}$$

By applying a $SU(4)$ transformation, as set out in Appendix A of [9], one can work in a gauge for which

$$\nu_2 = -x e_5 - y e_{12345} - c_1 e_{145} - c_2 e_{235} \ . \tag{A.16}$$

A.1.2 Null planes

We have already demonstrated above how to choose the two normal spinors (ν_1, ν_2) up to $SU(4) \ltimes \mathbb{R}^8$ transformations. The choice of the second spinor can be simplified further. For this, observe that if a direction in the space of the two normals (ν_1, ν_2) is associated with a time-like vector bilinear, then the corresponding background is a special case of those that will be investigated in section 6. Hence, it suffices to consider only those $SU(4) \ltimes \mathbb{R}^8$ cases for which all linear combinations of the two normals ν_1 and ν_2 are associated with null 1-form bilinears.

As we have shown above, there are two choices for the second normal given by

$$\nu_2 = -y e_{12345} - u^1 e_1 - u^2 e_2 - w e_{234} - c_3 e_{235} - c_4 e_{345} \ , \tag{A.17}$$

with $|u^1|^2 + |u^2|^2 \neq 0$, and

$$\nu_2 = -xe_5 - ye_{12345} - c_1 e_{145} - c_2 e_{235} \ . \tag{A.18}$$

For ν_2 given in (A.17), to impose the condition that the 1-form bilinear

$$\kappa_M = B(\nu_2 + k\nu_1, \Gamma_M C(\nu_2 + k\nu_1)^*) \tag{A.19}$$

is null for all k, we first compute κ^2 for $k = 0$ to find

$$\kappa^2 = -4\big((|u^1|^2 + |u^2|^2)(|y|^2 + |c_4|^2) + |c_3|^2|u^1|^2\big) \ . \tag{A.20}$$

This vanishes provided we take $y = c_4 = 0$, and either $c_3 = 0$ or $u^1 = 0$.

If $c_3 = c_4 = y = 0$ then the norm of κ, when $k = 1$, is given by $-4|u^2|^2|q|^2 - 4|\bar{w}p + u^1\bar{q}|^2$. Then, either $q = 0$ or $w = 0$, and one can make a $SU(4)$ gauge transformation to set

$$\nu_1 = e_5, \qquad \nu_2 = ce^1 \tag{A.21}$$

for $c \neq 0$, or $q \neq 0$, $u^2 = 0$, $u^1 \neq 0$ and $\bar{w}p + u^1\bar{q} = 0$. Thus one finds

$$\nu_2 = -ye_{12345} - u^1 e_1 - we_{234} \ . \tag{A.22}$$

Note that in this case, $w \neq 0$.

If, however, $c_4 = y = u^1 = 0$, then the norm of κ, when $k = 1$, is given by $-4|u^2|^2|q|^2 - 4|w|^2|p|^2$. Requiring this to vanish forces $q = w = 0$ and so

$$\nu_2 = -u^2 e_2 - c_3 e_{235} \ . \tag{A.23}$$

However, this normal is gauge equivalent to $\nu_2 = ce_1$ under an appropriately chosen $SU(4) \ltimes \mathbb{R}^8$ gauge transformation.

Hence, requiring that all linear combinations of ν_1, ν_2 generate null 1-forms reduces (A.17) to two simpler sub-cases, with either

$$\nu_1 = e_5, \qquad \nu_2 = ce^1 \qquad (c \neq 0) \tag{A.24}$$

or

$$\nu_1 = -pe_5 - qe_{12345}, \qquad \nu_2 = -ye_{12345} - u^1 e_1 - we_{234}, \tag{A.25}$$

with $\bar{w}p + u^1\bar{q} = 0$ and non-vanishing p, w, q, u^1.

It should be noted that for the case of ν_2 given in (A.18), all linear combinations of ν_1, ν_2 automatically generate null 1-forms, with no additional constraints on the coefficients in the normals.

A.2 $Spin(7) \ltimes \mathbb{R}^8$

The $Spin(7) \ltimes \mathbb{R}^8$ case is a special case of the $SU(4) \ltimes \mathbb{R}^8$ one. An inspection of section 4 for $G \neq 0$ reveals that, for all cases that $Spin(7) \ltimes \mathbb{R}^8$ arises as a special case of $SU(4) \ltimes \mathbb{R}^8$, the normal spinors of the former can be chosen as

$$\nu_1 = e_5 + e_{12345} \ , \quad \nu_2 = x(e_5 - e_{12345}) + c(e_{145} + e_{235}) \ , \tag{A.26}$$

where x, c are complex functions. The choice of normals can be further simplified. For this, choose a basis in the space of Killing spinors normal to (ν_1, ν_2) as

$$(\eta_a) = \{e_{15}, e_{25}, e_{35}, e_{45}, e_{1235}, e_{1245}, e_{1345}, e_{2345}, e_{12}, e_{13}, e_{24}, \\ e_{34}, e_{23} - e_{14}, c(1 - e_{1234}) - x(e_{23} + e_{14})\} \ . \tag{A.27}$$

Substituting this into the integrability condition (3.4), one finds that xc^* is a *real* valued function. Hence, without loss of generality, we can set

$$\nu_2 = e^{i\theta}(\rho_1(e_5 - e_{12345}) + \rho_2(e_{145} + e_{235})) \ , \tag{A.28}$$

where θ, ρ_1, ρ_2 are *real* functions. Using the gauge transformation $e^{\phi(\Gamma_{14}+\Gamma_{23})}$, where the gamma matrices are in the real basis and so ϕ is real, one can set $\rho_2 = 0$. Therefore, the two normal spinors can be chosen as

$$\nu_1 = e_5 + e_{12345}, \quad \nu_2 = c(e_5 - e_{12345}), \qquad (c \neq 0) \ . \tag{A.29}$$

Appendix B Gravitino Integrability condition

The integrability condition of the KSE is

$$[\mathcal{D}_N, \mathcal{D}_M]\epsilon \equiv \mathcal{R}_{NM}\epsilon = 2\mathcal{S}\epsilon - 2\mathcal{T}C\epsilon^* \tag{B.1}$$

where

$$\begin{aligned}
\mathcal{S} \ = \ & \frac{1}{8}R_{NM}{}^{L_1L_2}\Gamma_{L_1L_2} + \frac{i}{48}\Gamma^{L_1\dots L_4}D_{[N}F_{M]L_1\dots L_4} \\
&+\frac{1}{24}(-\Gamma^{L_1L_2}F_{[N|L_1}{}^{Q_1Q_2Q_3}F_{|M]L_2Q_1Q_2Q_3} + \frac{1}{2}\Gamma^{L_1\dots L_4}F_{NML_1}{}^{Q_1Q_2}F_{L_2L_3L_4Q_1Q_2} \\
&\qquad + \frac{1}{2}\Gamma_{[N}{}^{L_1L_2L_3}F_{M]L_1}{}^{Q_1Q_2Q_3}F_{L_2L_3Q_1Q_2Q_3}) \\
&+\frac{1}{32}(-\frac{1}{2}G_{[N}{}^{L_1L_2}G^\star_{M]L_1L_2} + \frac{1}{48}\Gamma_{NM}G^{L_1L_2L_3}G^\star_{L_1L_2L_3} \\
&\quad - \frac{1}{4}\Gamma_{[N}{}^{L_1}G_{M]}{}^{L_2L_3}G^\star_{L_1L_2L_3} + \frac{1}{8}\Gamma_{[N|}{}^{Q}G_Q{}^{L_1L_2}G^\star_{|M]L_1L_2} \\
&\quad + \frac{3}{16}\Gamma^{L_1L_2}G_{NM}{}^{L_3}G^\star_{L_1L_2L_3} - \Gamma^{L_1L_2}G_{[N|L_1}{}^{Q}G^\star_{|M]L_2Q} \\
&\quad - \frac{3}{16}\Gamma^{L_1L_2}G_{L_1L_2}{}^{Q}G^\star_{NMQ} + \frac{1}{16}\Gamma_{NM}{}^{L_1L_2}G_{L_1}{}^{Q_1Q_2}G^\star_{L_2Q_1Q_2} \\
&\quad - \frac{1}{16}\Gamma^{L_1\dots L_4}G_{L_1L_2L_3}G^\star_{NML_4} + \frac{1}{8}\Gamma_{[N|}{}^{L_1L_2L_3}G_{L_1L_2}{}^{Q}G^\star_{|M]L_3Q} \\
&\quad + \frac{1}{4}\Gamma^{L_1\dots L_4}G_{[N|L_1L_2}G^\star_{|M]L_3L_4} + \frac{1}{16}\Gamma^{L_1\dots L_4}G_{NML_1}G^\star_{L_2L_3L_4} \\
&\quad + \frac{1}{4}\Gamma_{[N|}{}^{L_1L_2L_3}G_{|M]L_1}{}^{Q}G^\star_{L_2L_3Q} + \frac{1}{24}\Gamma_{[N|}{}^{L_1\dots L_5}G_{|M]L_1L_2}G^\star_{L_3L_4L_5} \\
&\quad - \frac{1}{48}\Gamma_{[N|}{}^{L_1\dots L_5}G_{L_1L_2L_3}G^\star_{|M]L_4L_5} - \frac{1}{32}\Gamma_{NM}{}^{L_1\dots L_4}G_{L_1L_2}{}^{Q}G^\star_{L_3L_4Q} \\
&\quad - \frac{1}{288}\Gamma_{NM}{}^{L_1\dots L_6}G_{L_1L_2L_3}G^\star_{L_4L_5L_6}) \ ,
\end{aligned} \tag{B.2}$$

and

$$
\begin{aligned}
\mathcal{T} \;=\;& -\frac{1}{96}(\Gamma_{[N}{}^{L_1L_2L_3}D_{M]}G_{L_1L_2L_3} + 9\Gamma^{L_1L_2}D_{[N}G_{M]L_1L_2}) \\
& +\frac{i}{32}(\frac{1}{3}F_{NM}{}^{L_1L_2L_3}G_{L_1L_2L_3} + \Gamma^{L_1L_2}F_{[N|L_1L_2}{}^{Q_1Q_2}G_{|M]Q_1Q_2} \\
& \quad + \frac{1}{3}\Gamma_{[N}{}^{Q}F_{M]Q}{}^{L_1L_2L_3}G_{L_1L_2L_3} - \frac{1}{2}\Gamma^{L_1\dots L_4}F_{NML_1L_2}{}^{Q}G_{L_3L_4Q} \\
& \quad + \frac{1}{2}\Gamma_{[N}{}^{L_1L_2L_3}F_{M]L_1L_2}{}^{Q_1Q_2}G_{L_3Q_1Q_2} + \frac{1}{4}\Gamma^{L_1\dots L_4}F_{L_1\dots L_4}{}^{Q}G_{NMQ} \\
& \quad - \frac{1}{2}\Gamma_{[N|}{}^{L_1L_2L_3}F_{L_1L_2L_3}{}^{Q_1Q_2}G_{|M]Q_1Q_2}) \;.
\end{aligned} \tag{B.3}
$$

Appendix C Integrability condition

In this appendix, we shall solve the integrability condition (5.7)

$$(\hat{T}_{NM})_{L_1L_2L_3L_4}\Gamma^{L_1L_2L_3L_4}\eta_a = 0 \;, \tag{C.1}$$

for (η_a) given in (5.3) to show that $\nabla F = 0$, where

$$(\hat{T}_{NM})_{L_1L_2L_3L_4} = D_{[N}F_{M]L_1L_2L_3L_4} \;. \tag{C.2}$$

A straightforward but tedious calculation implies that all components of $\hat{T}$ are constrained to vanish, except for $(\hat{T}_{NM})_{\alpha_1\alpha_2\alpha_3\alpha_4}$ and $(\hat{T}_{NM})_{+\alpha_1\alpha_2\alpha_3}$ (and their complex conjugates) where α, β denote holomorphic indices in the standard holomorphic light-cone basis. In fact, these components also vanish. To see this, we make use of the conditions (4.29) on T^4. These imply in particular that

$$(\hat{T}_{[MN})_{L_1L_2L_3L_4]} = 0 \;, \tag{C.3}$$

$$(\hat{T}_{L_1(M})_{N)L_2L_3L_4} = (\hat{T}_{[L_1|(M})_{N)|L_2L_3L_4]} \;, \tag{C.4}$$

and

$$(\hat{T}_{M[N_1})_{N_2N_3N_4N_5]} = -\frac{1}{5!}\epsilon_{N_1N_2N_3N_4N_5}{}^{M_1M_2M_3M_4M_5}(\hat{T}_{M[M_1})_{M_2M_3M_4M_5]} \;. \tag{C.5}$$

Furthermore, as $F = e^+ \wedge \Phi$, and ce^+ is covariantly constant, it follows that

$$(\hat{T}_{\tilde{M}\tilde{N}})_{\tilde{L}_1\tilde{L}_2\tilde{L}_3\tilde{L}_4} = 0 \tag{C.6}$$

where $\tilde{N}$ and the other similar indices take all values except for “$+$”. This last property implies that

$$(\hat{T}_{\tilde{M}[\bar{\beta}})_{\mu_1\mu_2\mu_3\mu_4]} = 0 \;. \tag{C.7}$$

The self-duality of $\hat{T}$ on the anti-symmetrized indices implies

$$(\hat{T}_{\tilde{M}[+})_{-\lambda_1\lambda_2\lambda_3]} = 0 \tag{C.8}$$

and hence

$$(\hat{T}_{\tilde{M}-})_{+\lambda_1\lambda_2\lambda_3} = 0 \tag{C.9}$$

for $\tilde{M} = \alpha, \bar{\alpha}$. Next, observe that (C.4) implies that

$$(\hat{T}_{+\bar{\alpha}})_{+\alpha_1\alpha_2\alpha_3} = 0, \qquad (\hat{T}_{+-})_{+\alpha_1\alpha_2\alpha_3} = 0 \tag{C.10}$$

on symmetrizing on $\bar{\alpha}, \alpha_1$ and $-, \alpha_1$ respectively. Furthermore, (C.4) also implies that

$$(\hat{T}_{\bar{\alpha}\bar{\beta}})_{+\alpha_1\alpha_2\alpha_3} = 0 \tag{C.11}$$

on symmetrizing appropriately in $\bar{\alpha}, \alpha_1, \bar{\beta}, \alpha_2$. In addition, the self-duality condition (C.5) implies that

$$(\hat{T}_{M[+})_{\alpha_1\alpha_2\alpha_3\alpha_4]} = 0 \ . \tag{C.12}$$

On setting $M = +$ in (C.12) one finds

$$(\hat{T}_{+[\alpha_1})_{|+|\alpha_2\alpha_3\alpha_4]} = 0 \ . \tag{C.13}$$

However, (C.4) implies that $(\hat{T}_{+\alpha_1})_{+\alpha_2\alpha_3\alpha_4}$ is totally antisymmetric in α_1, α_2, α_3, α_4, and hence

$$(\hat{T}_{+\alpha_1})_{+\alpha_2\alpha_3\alpha_4} = 0 \ . \tag{C.14}$$

On setting $M = \bar{\beta}$ in (C.12), one finds the condition

$$(\hat{T}_{\bar{\beta}+})_{\alpha_1\alpha_2\alpha_3\alpha_4} - 4(\hat{T}_{\bar{\beta}[\alpha_1})_{|+|\alpha_2\alpha_3\alpha_4]} = 0 \ . \tag{C.15}$$

However, (C.4) implies that $(\hat{T}_{\bar{\beta}\alpha_1})_{+\alpha_2\alpha_3\alpha_4}$ is totally antisymmetric in α_1, α_2, α_3, α_4, and furthermore that

$$(\hat{T}_{\bar{\beta}+})_{\alpha_1\alpha_2\alpha_3\alpha_4} = -(\hat{T}_{\bar{\beta}\alpha_1})_{+\alpha_2\alpha_3\alpha_4} \ . \tag{C.16}$$

It follows that

$$(\hat{T}_{\bar{\beta}+})_{\alpha_1\alpha_2\alpha_3\alpha_4} = (\hat{T}_{\bar{\beta}\alpha_1})_{+\alpha_2\alpha_3\alpha_4} = 0 \ . \tag{C.17}$$

On setting $M = \alpha$ in (C.12), and noting that for a non-zero expression one can take without loss of generality $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ to be distinct, with $\alpha = \alpha_1$, it is straightforward to show that

$$(\hat{T}_{\alpha+})_{\alpha_1\alpha_2\alpha_3\alpha_4} = 0 \ , \tag{C.18}$$

where (C.4) has also been used. Next, consider $(\hat{T}_{\alpha_1\alpha_2})_{+\beta_1\beta_2\beta_3}$; without loss of generality one can take $\alpha_1 = \beta_1$, then on using (C.4) to symmetrize on the $+, \alpha_2$ indices, one finds

$$(\hat{T}_{\alpha_1\alpha_2})_{+\beta_1\beta_2\beta_3} = 0 \ . \tag{C.19}$$

Hence, we have shown $(\hat{T}_{MN})_{+\alpha_1\alpha_2\alpha_2} = 0$ for all M, N.

To proceed, note that (C.6) implies that

$$
\begin{aligned}
(\hat{T}_{\alpha_1\alpha_2})_{\beta_1\beta_2\beta_3\beta_4} &= 0, \qquad (\hat{T}_{\alpha\bar{\beta}})_{\beta_1\beta_2\beta_3\beta_4} = 0, \qquad (\hat{T}_{\bar{\alpha}_1\bar{\alpha}_2})_{\beta_1\beta_2\beta_3\beta_4} = 0, \\
(\hat{T}_{-\alpha})_{\beta_1\beta_2\beta_3\beta_4} &= 0, \qquad (\hat{T}_{-\bar{\alpha}})_{\beta_1\beta_2\beta_3\beta_4} = 0 \ ,
\end{aligned}
\tag{C.20}
$$

and on setting $M = -$ in (C.12) one also finds

$$
(\hat{T}_{-+})_{\beta_1\beta_2\beta_3\beta_4} \ . \tag{C.21}
$$

Hence $(\hat{T}_{MN})_{\alpha_1\alpha_2\alpha_3\alpha_4} = 0$ for all M, N; so $\hat{T} = 0$. In turn, this and the Bianchi identity for F imply that $\nabla F = 0$ as in the case of the maximally supersymmetric backgrounds in [1].

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